A CLASS OF CONTINUED FRACTIONS ASSOCIATED WITH CERTAIN PROPERLY DISCONTINUOUS GROUPS

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1. **Introduction.** This investigation stems from the importance of certain groups of linear fractional transformations in the study of Dirichlet series satisfying a functional equation.

By $\Gamma(\lambda)$ we shall mean the group of linear fractional transformations of the complex plane on itself,

$$z' = V(z) = (az + b)/(cz + d), \qquad ad - bc = 1, \tag{1.1}$$

with real coefficients, and generated by the transformations,

$$S = S(z) = z + \lambda, \qquad T = T(z) = -1/z, \qquad I = I(z) = z, \tag{1.2}$$

where λ is a fixed positive real number. These groups are useful in the study of Dirichlet series only when $\Gamma(\lambda)$ is properly discontinuous, that is a Fuchsian group. It has been shown by E. Hecke [4], that $\Gamma(\lambda)$ is Fuchsian if and only if $\lambda = 2 \cos \pi/q$, $q =$ integer ≥ 3, when $\lambda < 2$; and for every real λ when $\lambda > 2$. The symbols $\Gamma_1(\lambda)$ and $\Gamma_2(\lambda)$ shall denote a Fuchsian group when $\lambda < 2$ and $\lambda > 2$ respectively. $\Gamma(\lambda)$ will now mean the totality of Fuchsian groups, namely $\Gamma(\lambda) = \Gamma_1(\lambda) + \Gamma_2(\lambda)$. The cases $\lambda = 1$, the modular group, and $\lambda = 2$, a subgroup of the modular will be omitted from our discussion.

It has been further established by Hecke [4], that a standard fundamental region (FR) of the group is the domain in the upper half of the complex plane defined by: $-\lambda/2 \leq R(z) \leq \lambda/2$, and $|z| \geq 1$, with the real axis considered as the principal circle. From the theory of automorphic functions (see for example, [1] or [2]), we find that the groups $\Gamma(\lambda)$ have the following important property: For $\Gamma_1(\lambda)$, the set of limit points on the principal circle is a perfect everywhere dense set of points. For $\Gamma_2(\lambda)$, the set of limit points is a perfect nowhere dense set. Moreover, from the nature of the FR the generators of the group defined by (1.2) satisfy the relations,

$$\begin{aligned} &T^2 = I, \qquad (TS)^q = TSTS \cdots TS = (S^{-1}T)^q = I; \qquad &(\lambda < 2) \\ &T^2 = I; \qquad &(\lambda > 2). \end{aligned} \tag{1.3}$$

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