POINT SYSTEMS FOR LAGRANGE INTERPOLATION

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1. **Introduction.** The n-th Lagrange interpolation polynomial for any single-valued real function $f(x)$ defined on the closed interval $[-1, 1]$ is the unique polynomial of degree $(n - 1)$ coinciding with $f(x)$ at the distinct fundamental points

$$(1.1) \qquad -1 < x_n^n < x_{n-1}^n < \cdots < x_1^n < 1, \qquad x_{n+1}^n = -1, \qquad x_0^n = 1.$$

This polynomial is given by the formula

$$(1.2) \qquad L_n(f) = \sum_{i=1}^{n} f(x_i^n) l_i^n(x)$$

with

$$(1.3) \qquad l_i^n(x) = \frac{w_n(x)}{w_n'(x_i^n)(x - x_i^n)}, \qquad w_n(x) = \prod_{i=1}^{n} (x - x_i^n).$$

If $P_n(x)$ is any polynomial of degree less than n, we have $L_n(P_n) \equiv P_n$ and in the interval $[-1, 1]$,

$$(1.4) \qquad \begin{aligned} | L_n(f) - f(x) | &= | L_n(f) - L_n(P_n) + P_n - f(x) | \\ &\leq (\max_{[-1,1]} | f - P_n |)\left(1 + \sum_{i=1}^{n} | l_i^n(x) |\right). \end{aligned}$$

The Lebesgue function will be denoted by $E_n(x)$, i.e.,

$$(1.5) \qquad E_n(x) = \sum_{i=1}^{n} | l_i^n(x) |.$$

From (1.4) we see that if the order of the term $| f - P_n |$ is known, then the order of the Lebesgue function is all that is needed to determine the degree of approximation of f given by $L_n(f)$. The best uniform order that can be obtained for the Lebesgue function for any point system is $\log n$ as shown in the thorough investigations of S. Bernstein [1; 1025] and H. Hahn [4]. In particular, S. Bernstein [1; 1027] not only shows that the zeros of the Tchebichef polynomials $P_n(x) = \cos n\theta$, with $x = \cos \theta$, yield a point system with a Lebesgue function of uniform order $\log n$ in $[-1, 1]$, but he also derives other point systems with Lebesgue functions of the same order by "distorting" the Tchebichef point system in a particular manner. We shall now obtain a class of point systems in the interval $[-1, 1]$ which includes the Tchebichef system, and such that not

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