COMMUTATIVE SEMIGROUPS

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1. **Introduction.** The development of an "ideal theory" for commutative semigroups with the cancellation law has been investigated (see [1] and [3]) with the aim of obtaining analogues of the multiplicative ideal theory of commutative rings. It appears to have escaped notice, however, that the most general structure theory for commutative rings is almost purely a theory of multiplication (see [2; 1-10, 15-17, 20-21] or [4; 22-42]). That is to say, it treats the ring as a commutative semigroup. It is the purpose of this paper to develop this theory as directly as possible in so far as it applies to general commutative semigroups.

It would have been possible to adhere closely to the organization of ideal theory by suitably generalizing the notion of ideal. While this procedure would have made clearer the origins of various parts of the theory, it would have obscured its essential features. In spite of considerable reorganization the ideal theory of commutative rings has not been lost. It remains the principal domain of application of the results of this paper.

2. **Preliminaries.** Let $\mathfrak{M}$ be a commutative semigroup, that is, $\mathfrak{M}$ is closed with respect to an associative and commutative binary operation which we shall denote by juxtaposition. The usual notational conventions of group theory will be used together with others to be defined. Let S be a subset of $\mathfrak{M}$. The set of those elements s of S for which $s, s^2, s^3, \cdots$ are all contained in S will be called the co-radical of S and will be denoted by $S^{\blacktriangle}$. S will be called a q-set or a p-set according as $SS^{\blacktriangle} \subset S$ or $SS \subset S$. If T is also a subset of $\mathfrak{M}$ then S: T shall denote the set of all elements a of $\mathfrak{M}$ such that $a\,T \subset S$. If T is empty then $S{:}T = \mathfrak{M}$.

A subset S of $\mathfrak{M}$ will be called a co-ideal if $ab \in S$ implies $a, b \in S$. The empty set will be regarded as one of the co-ideals of $\mathfrak{M}$. The following frequently used statements are immediate consequences of this definition.

(a) *If S is a co-ideal and $a \notin S$ then $a\mathfrak{M}$ and S are disjoint.*

(b) *The union of the members of a family of co-ideals is a co-ideal.*

(c) *The co-radical of a co-ideal is a co-ideal.*

(d) *If S is a co-ideal and T is a non-empty subset of $\mathfrak{M}$ then $S{:}T$ is a co-ideal contained in S which is non-empty only if $T \subset S$.*

A co-ideal which is a q-set will be called *primary*. If it is a p-set then it will be called *prime.*

(e) *The co-radical of a primary co-ideal is prime.*

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