SOME THEOREMS ON BERNOULLI AND EULER NUMBERS OF HIGHER ORDER

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1. Introduction. Put

$$(1.1)\qquad \begin{cases} \left(\dfrac{x}{e^x - 1}\right)^k = \displaystyle\sum_{m=0}^{\infty} B_m^{(k)} \frac{x^m}{m!} & (B_m = B_m^{(1)}), \\ \left(\dfrac{2}{e^x + 1}\right)^k = \displaystyle\sum_{m=0}^{\infty} \frac{C_m^{(k)}}{2^m} \frac{x^m}{m!} & (C_m = C_m^{(1)}). \end{cases}$$

A well-known theorem of Glaisher [4; 325] asserts (in different notation) that

$$(1.2)\qquad \begin{cases} B_{2r}^{(p)} \equiv -\dfrac{1}{2r} pB_{2r} & (\text{mod } p^2) \\ B_{2r+1}^{(p)} \equiv \dfrac{2r+1}{4r} p^2 B_{2r} & (\text{mod } p^3), \end{cases}$$

where $1 \leq r \leq \frac{1}{2}(p - 3)$ and p is a prime > 3. Nielsen [5; 338], also using different notation, has proved that

$$(1.3)\qquad \begin{cases} B_{2r}^{(-p)} \equiv \dfrac{1}{2r} pB_{2r} & (\text{mod } p^2) \\ B_{2r+1}^{(-p)} \equiv \dfrac{2r+1}{4r} p^2 B_{2r} & (\text{mod } p^3), \end{cases}$$

where again $1 \leq r \leq \frac{1}{2}(p - 3)$. As for the $C_m^{(k)}$ we have

$$(1.4)\qquad \begin{cases} C_{2r}^{(p)} \equiv pC_{2r-1} & (\text{mod } p^2) \\ C_{2r+1}^{(p)} \equiv -(2r+1)p^2 C_{2r-1} & (\text{mod } p^3) \end{cases}$$

and

$$(1.5)\qquad \begin{cases} C_{2r}^{(-p)} \equiv -pC_{2r-1} & (\text{mod } p^2) \\ C_{2r+1}^{(-p)} \equiv -(2r+1)p^2 C_{2r-1} & (\text{mod } p^3), \end{cases}$$

where now $r \geq 1$. The result (1.5) is due to Nielsen [5; 292]. The formulas (1.2), (1.3), (1.4), (1.5) are proved in a uniform manner in [2].

Nörlund [6; Chapter 6] has defined more general numbers $B_m^{(k)}[\omega_1, \cdots, \omega_k]$,

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