GENERALIZATION OF THE RECIPROCITY FORMULA FOR DEDEKIND SUMS

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Recently Mordell [2] has proved a certain relation in which three Dedekind sums appear together and which includes the well-known reciprocity formula as a special case. However, Mordell's formula contains as a further summand the number of lattice points in a certain tetrahedron and therefore has to be looked upon rather as a determination of this number of lattice points by means of Dedekind sums. A method developed by Redei [4] and simplified by Carlitz [1] leads now to another relation between 3 Dedekind sums, a formula which contains otherwise only elementary expressions, and which can be looked upon as a natural generalization of the reciprocity formula, which it implies as a special case. The formula in question seems to go beyond those properties which express the group property of the modular substitution in its application to log $\eta(t)$. Since Redei's and Carlitz's notation differ from mine I shall briefly derive the basic formulae again.

1. With the notation

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{for } x \text{ not integer} \\ 0 & \text{for } x \text{ integer} \end{cases}$$

the function $((\mu/a))$ is periodic in μ of period a, where $a > 0$ and μ are integers. With ξ a primitive a-th root of unity there exists therefore a "finite Fourier expansion"

$$\left(\left(\frac{\mu}{a}\right)\right) = \sum_{j=0}^{a-1} A_j \xi^{\mu j},$$

where

$$A_j = \frac{1}{a} \sum_{\mu=1}^{a} \left(\left(\frac{\mu}{a}\right)\right) \xi^{-j\mu}$$

and in particular

$$A_0 = \frac{1}{a} \sum_{\mu=1}^{a} \left(\left(\frac{\mu}{a}\right)\right) = 0,$$

and for $0 < j < a$

$$A_j = \frac{1}{a} \sum_{\mu=1}^{a-1} \left(\frac{\mu}{a} - \frac{1}{2}\right) \xi^{-j\mu} = \frac{1}{a} \frac{1}{\xi^{-j} - 1} + \frac{1}{2a},$$

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