POWER SERIES AND PEANO CURVES

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It is well known that there exist real continuous functions $F(\theta)$, $G(\theta)$ such that the curve $x + iy = F(\theta) + iG(\theta)$ fills a square. Such a locus is called a Peano curve. It is a surprising result of Salem and Zygmund [1] that a Peano curve may be the boundary values of a function which is regular in the unit circle. They defined a class of lacunary series

$$f(z) = \sum_{\nu=1}^{\infty} a_\nu z^{\lambda_\nu}$$

which are regular in $| z | < 1$, continuous in $| z | \leq 1$, and such that the values of $f(e^{i\theta})$ cover some square.

In the present note it is shown that there are non-constant functions $f(z)$ regular in $| z | < 1$ and continuous in $| z | \leq 1$ such that $f(z)$ takes on the unit circumference every value which it assumes in $| z | < 1$. As might be expected, the conditions imposed on $f(z)$ to obtain this stronger property are more stringent than those used by Salem and Zygmund.

Theorem. *Let*

$$f(z) = \sum_{\nu=1}^{\infty} a_\nu z^{\lambda_\nu} \qquad (a_1 \neq 0),$$

where $\sum | a_\nu |$ *converges,*

$$\left| \frac{a_{\nu+1}}{a_\nu} \right| \geq \rho > \tfrac{3}{4}, \qquad \frac{\lambda_{\nu+1}}{\lambda_\nu} \geq \mu_\nu \tag{2}$$

and $\mu_1, \mu_2, \cdots$ *is an increasing sequence such that*

$$\sum_{\nu=1}^{\infty} \frac{1}{(\mu_\nu)^{1/2}} \leq \frac{1}{10} . \tag{3}$$

Then $f(e^{i\theta})$ *takes every value which* $f(z)$ *assumes in* $| z | \leq 1$.

It is clear that the function $f(z)$ thus defined is regular in $| z | < 1$ and continuous in $| z | \leq 1$. The gap condition stated in (3) is much more severe than in the usual gap theorem. It would be interesting to determine whether (3) can be replaced by a condition $\mu_\nu \geq \mu$ for some constant μ. The proof is elementary, but some preliminary results are obtained before turning to the theorem.

Let the hypotheses of the theorem be satisfied and define $s_0(\theta) = 0$,

$$s_n(\theta) = \sum_{\nu=1}^{n} a_\nu \exp(i\lambda_\nu \theta) \qquad (n \geq 1).$$

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