CONTINUITY OF TRANSFORMATION GROUPS IN TOPOLOGICAL SPACES

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Introduction. Let X be an abstract group topologized in such a way that the group translations are continuous. X is a homogeneous space. Conditions under which X becomes a topological group have been given by Montgomery for the case where X is a metric space. We shall consider a T_1 topological space X which is homogeneous under an Abelian group G of homeomorphisms of X and shall find first a necessary and sufficient condition, condition A, that X become a topological group with G as the group of translations of X. Examples will be given to show that even in the case where X is a uniform or a metric space condition A may not be omitted.

Next, letting X be an abstract group with a topology, we will assume that for any integers p and q and any $x \in X$ there exists a unique $y \in X$ such that $qy = px$ where $px = x + x + \cdots + x(p \text{ times})$. Thus X is a linear space over the rationals. We shall obtain a sufficient condition for the continuity of rx, r rational, $x \in X$, in r and x simultaneously, assuming continuity in r and x separately.

Finally, a metric shall be introduced in X and assumptions made about the metric which will enable us to imbed X in a Banach space.

1. We begin with the following definitions:

Definition 1. A topological space X is said to be *homogeneous* if there exists a group G of homeomorphisms of X onto itself such that given any two points x and y of X, there exists a $T \in G$ such that $T(x) = y$.

Definition 2. X is called *strictly homogeneous* under the group G if the $T \in G$ is unique.

When G is Abelian we have the following relation between homogeneity and strict homogeneity.

Lemma. *If X is a homogeneous T_1 space and G is Abelian, then X is strictly homogeneous under G.*

Proof. If $T(e) = x$ and $T'(e) = x$ where $T, T' \in G$ and $e, x \in X$, then for any $y \in X$ there is a $T'' \in G$ such that $T''(e) = y$ and so $T''T(e) = T''T'(e)$. Now $TT''(e) = T'T''(e)$ so that $T(y) = T'(y)$ for any y. Therefore $T = T'$.

Definition 3. A topological space X homogeneous under a group G of homeomorphisms of X is said to satisfy *condition* A if given any open set O

Received May 4, 1953. This paper is a portion of a dissertation accepted by Bryn Mawr College in partial fulfillment of the requirements for the Ph.D. degree.