COVERING SPACES, FIBRE SPACES, AND LOCAL HOMEOMORPHISMS

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Let X and Y be topological spaces, f a continuous mapping of X into Y. We shall say that (X, Y, f) is a covering triple if X is a covering space of Y with f as covering mapping. It is the purpose of this paper to characterize covering triples from among triples satisfying weaker structural conditions. In §1, it is shown under mild local connectedness conditions on X and Y, that covering spaces can be distinguished among the fibre spaces of various types, weak and strong, by the pathwise total disconnectedness of the fibres. In §§2 and 3 it is established that with suitable local connectedness conditions on Y, a local homeomorphism f of X into Y is a covering mapping provided that there exists for each point y of Y a neighborhood V such that f is a closed mapping of each component of $f^{-1}(V)$ into V. As a special case, it follows that every local homeomorphism of X into Y which is a closed mapping is also a covering mapping.

The results of this paper were obtained in connection with a study of the rigorous application of the monodromy argument in function spaces to obtain results on the uniqueness and multiplicity of solutions of non-linear functional equations. The author is indebted to Professor W. Hurewicz for a number of stimulating conversations on the questions treated here.

1. Let X and Y be two Hausdorff spaces, f a continuous mapping of X into Y. X is said to be a covering space of Y with f as covering mapping provided that f maps X onto Y and that for each point y of Y there exists a neighborhood V such that $f^{-1}(V)$ is the union of a pairwise disjoint family of open sets of X, each of which is mapped homeomorphically onto V by f. (For the definition and properties of covering spaces, compare [7] and [9]. Another definition which is equivalent to the above when Y is locally connected is given in [2].)

We shall consider two different classes of fibre spaces, one of which includes the other. The strong fibre spaces are the fibre spaces of Hurewicz and Steenrod [6] as generalized by Hu [5] to include the fibre bundles. The class of weak fibre spaces is defined by the requirement that the covering homotopy theorem hold for mappings of line segments and includes the class of strong fibre spaces.

X is a strong fibre space over Y with fibre mapping f if f maps X onto Y and if for each point y_0 of Y there exists a neighborhood V of y_0 and a continuous mapping ϕ_V of $V \times f^{-1}(V)$ into X such that,

$$(1.1) \qquad (f \circ \phi_V)(y, x) = y \qquad (y \in V,\ x \in f^{-1}(V)).$$

$$(1.2) \qquad \phi_V(f(x), x) = x \qquad (x \in f^{-1}(V)).$$

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