THE LEBESGUE CONSTANTS FOR EULER (E, p) SUMMATION OF FOURIER SERIES

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1. **Introduction.** The regular Hausdorff method of summability $H(g)$ associates with a given sequence $\{s_k\}_0^\infty$ the means

$$t_n = \sum_{k=0}^{n} {}_nC_k s_k \int_0^1 t^k(1 - t)^{n-k}\, dg(t),$$

where $g(t)$ is of bounded variation on the interval $0 \leq t \leq 1$, $g(0+) = g(0)$, and $g(1) - g(0) = 1$. The Lebesgue constant of order n for the method $H(g)$ is then defined to be

$$L(n; g) = \frac{1}{\pi} \int_0^{2\pi} \left| \int_0^1 \operatorname{Im} \{e^{iu/2}(1 - t + te^{iu})^n\}\, dg(t) \right| \frac{du}{2 \sin u/2},$$

where Im $\{w\}$ denotes the imaginary part of the complex number w.

It is well known and easy to see that if $L(n; g) \to \infty$ as $n \to \infty$, then there is a continuous function $f(y)$ whose Fourier series is not summable $H(g)$ for at least one value of y. It is therefore of some interest to know the asymptotic behavior of $L(n; g)$ as $n \to \infty$.

If $g(t)$ is the characteristic function $E_r(t)$ of the closed interval $r \leq t \leq 1$, $0 < r \leq 1$, then the method $H(E_r)$ is ordinarily denoted by $(E, (1 - r)/r)$ and is said to be an Euler summability method. Lee Lorch has shown [1] that

$$L(x; E_{1/2}) = \frac{2}{\pi^2} \log 2x + A + O(x^{-1/2})$$

as $x \to \infty$, where x is a continuous parameter and

$$(1) \qquad A = -\frac{2}{\pi^2} C + \frac{2}{\pi} \int_0^1 u^{-1} \sin u\, du - \frac{2}{\pi} \int_1^\infty u^{-1} \left\{ \frac{2}{\pi} - |\sin u| \right\} du,$$

C being the Euler-Mascheroni constant.

It is the purpose of this note to show that

$$(2) \qquad L(n; E_r) = \frac{2}{\pi^2} \log \frac{2nr}{1 - r} + A + \epsilon_n(r)$$

for $0 < r < 1$, where A is defined by (1) and $\epsilon_n(r) \to 0$ as $n \to \infty$. This will be effected by showing that

$$L(n; E_r) = L(nr/(1 - r); E_{1/2}) + o(1)$$

as $n \to \infty$.

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