THE SOLUTIONS OF THE EULER-POISSON-DARBOUX EQUATION FOR NEGATIVE VALUES OF THE PARAMETER

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1. **Introduction.** The hyperbolic differential equation

$$E(k) : \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} - \frac{k}{t}\frac{\partial u}{\partial t} = 0$$

is referrde to as the Euler-Poisson-Darboux equation. The solution of a singular Cauchy problem for this equation for an arbitrary real value of the parameter k has been the subject of recent research. The uniqueness of such a solution for $k > 0$ was established by Ásgeirsson [1] using the method of Zaremba [6]. For $k < 0$, as Weinstein [3], [4] has pointed out, the solution is not unique. Accordingly, there is the problem of determining all solutions for a given negative k. This is the objective of the present paper.

2. **Preliminary remarks.** The Cauchy problem with which we are concerned can be formulated as follows. Let (B, C) be an open interval of the x-axis. Let T be a characteristic triangle in the (x, t)-plane formed by the closed segment $[B, C]$ and the characteristics which pass through points B and C. (Note that the characteristics are straight lines having slopes 1 and -1.) For definiteness, we choose T to be the triangle having its third vertex, A, in the upper half-plane $(t > 0)$. Further, let G be a region which contains at least the points in the interior of T as well as those on sides AB and AC. However, G need not contain the points of the closed interval $[B, C]$.

A solution, $u^{(k)}$, of $E(k)$ is "regular in T" if there is a region G of the above type in which $u^{(k)}$ has continuous second derivatives. No stipulation is made about the behavior of the function or its derivatives on the interval $[B, C]$.

Finally, suppose $f(x)$ is an arbitrary function having continuous derivatives of sufficiently high order (to be specified precisely below) on the interval (B, C). By a solution of the Cauchy problem for $E(k)$ we mean a solution, $u^{(k)}(x, t)$, of the equation which is regular in T and such that (1) $\lim_{t \to 0^+} u^{(k)}(x, t) = f(x)$ and (2) $\lim_{t \to 0^+} u_t^{(k)}(x, t) = 0$.

Before we consider the Cauchy problem for $k < 0$, we shall obtain an integral representation of an arbitrary solution of $E(k)$. To do this we shall use two lemmas, the proofs of which are to be found in [2].

Lemma 2.1. *If $u^{(k+2)}$ is a solution of $E(k + 2)$ which is regular in T, then there is a solution $u^{(k)}$ of $E(k)$ which is also regular in T and such that $\partial u^{(k)}/\partial t = tu^{(k+2)}$. Further, the third derivatives of $u^{(k)}$ are continuous in a region, G, of the kind described above.*

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