AUTOMORPHS OF QUADRATIC FORMS

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1. **Introduction.** Probably the most elegant expression for the automorphs of a symmetric matrix is Cayley's [1], [6; 65, 66]. But his form does not seem to lend itself to the determination of integral automorphs, that is, automorphic transformations whose elements are rational integers. In the case of ternary forms, Hermite, Bachmann [2] and others obtained general expressions for the automorphs. More recently Peter Scherk [7] showed that every automorph could be expressed as a product of a definite number of so-called "symmetries" and Sominskiĭ [8] found all ternary automorphs whose square is the identity. Of all these results, the forms of Hermite and the results of Sominskiĭ seem to bear the greatest promise of giving manageable forms for the automorphs.

In this paper we define $L = T - T^I$, show that $L = RA/d$, where d is the determinant of A and R is a skew matrix, and then proceed to express T as a polynomial in L whenever such an expression exists. We find explicit conditions on the characteristic function of T that such a polynomial expression shall exist. Our expressions particularize to Hermite's for three variables but in this case we derive a more manageable form. Throughout this paper the matrix A is assumed to be symmetric. Except where otherwise specified the elements of all matrices are in an arbitrary field of characteristic different from 2.

When L is zero, it is obvious that T cannot be expressed as a polynomial in L and in §5 we find all automorphs T for which $L = 0$, that is, for which $T^2 = I$, thus generalizing Sominskiĭ's results. In all cases there seems to be considerable simplification resulting from the use of modern matrix methods.

Thus, let T be an automorph of a symmetric matrix A, that is, $T^TAT = A$.

We first prove

Theorem 1. *If T is an automorph of A, and we let $L = T - T^I$, then for some skew matrix R, $L = RA/d$, where d is the determinant of A.*

Proof. Let

$$\overline{A} = QT^T \tag{1}$$

define a matrix Q, where $\overline{A}$ is the adjoint matrix of the symmetric matrix A. From (1) we have $Q = \overline{A}T^{TI}$ and hence

$$Q - Q^T = \overline{A}T^{TI} - T^I\overline{A}.$$

But $T^TAT = A$ implies $T^I\overline{A}T^{TI} = \overline{A}$ and hence

$$Q - Q^T = (T - T^I)\overline{A}. \tag{2}$$

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