LOCAL PROPERTIES OF TOPOLOGICAL SPACES

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Introduction. Let us say that a topological space X has a certain property $\mathfrak{P}$ *locally* if every element of X has a neighborhood (not necessarily open) which has property $\mathfrak{P}$. It is immediately obvious that any X which has property $\mathfrak{P}$ must also have property $\mathfrak{P}$ locally. But some properties $\mathfrak{P}$, such as local compactness or local connectedness, clearly also satisfy the following converse (for any X):

(1) If X has property $\mathfrak{P}$ locally, then X has property $\mathfrak{P}$.

The purpose of this paper is to show that (1), and a related assertion, also hold in many cases where this is not at all obvious. We will do so by proving two theorems (Theorem 3.6 and Theorem 5.5) which give very general sufficient conditions for the validity of (1), and we will then apply these theorems to some special cases.

The most effective method of attacking our problem seems to be by means of the normal coverings of Tukey [10; 46], which are defined and discussed in §2. Our two theorems (Theorem 3.6 and Theorem 5.5) are then proved in §§3 and 5, and some of their consequences are obtained in §§4 and 6.

The general approach of this paper was suggested by the proof of a theorem of Hanner [4; Theorem 3.3], and one of our results is a generalization of this theorem (compare Proposition 4.1).

2. **Normal coverings.** Let us begin this section by recalling the definitions of some of the terms which will be used below. Let X be a topological space. A *covering* of X is a collection of open subsets of X whose union is X. If $\mathfrak{U}$ and $\mathfrak{V}$ are coverings of X, then $\mathfrak{V}$ is a *refinement* of $\mathfrak{U}$ if every member of $\mathfrak{V}$ is a subset of some member of $\mathfrak{U}$; $\mathfrak{V}$ is a Δ-*refinement* [10; 45] of $\mathfrak{U}$ if, for every x in X, $\bigcup \{V \varepsilon \mathfrak{V} \mid x \varepsilon V\}$ is a subset of some member of $\mathfrak{U}$. A covering $\mathfrak{U}$ of X is *normal* [10; 46] if there exists a sequence $\{\mathfrak{U}_n\}_{n=0}^{\infty}$ of coverings of X such that $\mathfrak{U} = \mathfrak{U}_0$, and $\mathfrak{U}_{n+1}$ is a Δ-refinement of $\mathfrak{U}_n$ for $n = 0, 1, \cdots$. A covering $\mathfrak{U}$ is called *locally finite* [3; 66] if every x in X has a neighborhood which intersects only finitely many members of $\mathfrak{U}$. The space X is *paracompact* [3; 66] if it is Hausdorff, and if every covering of X has a locally finite refinement. (A. H. Stone showed [9; Theorem 1] that a Hausdorff space Y is paracompact if and only if every covering of Y is normal). A collection $\mathfrak{U}$ of subsets of X is called *discrete* [1; 176] if the closures of the members of $\mathfrak{U}$ are disjoint, and if every subcollection of these closures has a closed union; it is called σ-*discrete* if it is the union of countably many discrete subcollections. A *pseudometric* [10; 50] on X is a function ρ from $X \times X$ to the non-negative real numbers such that

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