NON-COMMUTATIVE CYCLIC FIELDS

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The structure of cyclic extensions of commutative fields was completely determined by the works of Artin-Schreier [4], A. A. Albert[1] (see also [2]) and Witt [17]. The object of the present paper is to extend these results to the non-commutative case. The case of quadratic extensions was recently dealt with by Dieudonné [6].

Throughout this paper we denote by $\mathfrak{F}$ and $\mathfrak{C}$ quasi-fields with centers Δ and Γ. We call $\mathfrak{F}$ a *cyclic extension* of $\mathfrak{C}$ (or $\mathfrak{F}$ is cyclically extended from $\mathfrak{C}$) if $\mathfrak{F}$ posses a cyclic group $\mathfrak{G}$ of n automorphisms with $\mathfrak{C}$ the set of all fixed elements, and $\mathfrak{F}$ is a right $\mathfrak{C}$-module of dimension n (we use the notation $(\mathfrak{F}:\mathfrak{C})_r = n$). The last requirement is superfluous for commutative fields $\mathfrak{F}$ or for groups of outer automorphisms (except the identity). But it is essential in developing the structure of non-commutative cyclic extensions as will be shown in §1 by an example which is due to the referee. A generally applicable condition that $(\mathfrak{F}:\mathfrak{C})_r = n$ is given in [8].

The following are simple methods for obtaining cyclic extensions of quasi-fields.

(1) Let $\alpha \in \mathfrak{F}$ satisfy an irreducible equation $x^n - \beta = 0$ in Δ, then the inner automorphism: $a \rightarrow \alpha a \alpha^{-1}$ in $\mathfrak{F}$ generates a cyclic group of automorphisms of $\mathfrak{F}$ of order n. Let $\mathfrak{C}$ be the centralizer of α in $\mathfrak{F}$, then by results of [5] $(\mathfrak{F}:\mathfrak{C})_r = (\Delta(\alpha):\Delta) = n$ which means that $\mathfrak{F}$ is a cyclic extension of order n of $\mathfrak{C}$. Furthermore, the results of [5] show that this is always the case if the group $\mathfrak{G}$ contains only inner automorphisms. A situation of this type one meets in forming cyclic division algebras over Δ.

(2) Let $\mathfrak{F} = \mathfrak{C} \times \Delta$ (over Γ), where Δ is now a commutative cyclic extension of Γ of the same order. Let $\mathfrak{G}$ be a group of extensions (of the same order) of the automorphisms of Δ over Γ. By [7] it follows that this is always the case if $(\mathfrak{F}:\Delta) < \infty$. For another approach see [12] and [13].

We shall refer to extensions of the preceding types, or combinations of them as trivial extensions. Note that for non-trivial cyclic extensions $(\mathfrak{F}:\Delta)$ cannot be finite and the group $\mathfrak{G}$ must contain some outer automorphisms. An example of a non-trivial cyclic extension was given by Köthe in [11; 24] (quoted in [7]); additional examples, for any n, of a different type will be given in §2.

As in the commutative case, the study of cyclic extensions must be carried out in a parallel fashion for quasi-fields of characteristic $p \neq 0$ with extensions of degree $n = p^e$, and for characteristic zero or extensions for which $(n, p) = 1$. The commutative extensions of degree p, in case of characteristic p, are obtained by adjoining a root of an irreducible equation of the type $x^p - x - a = 0$.

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