AN EXISTENCE THEOREM FOR THE GENERALIZED TRICOMI PROBLEM

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1. In a recent paper Frankl [2] considered a new boundary value problem for the Tricomi equation

$$Lu \equiv yu_{xx} + u_{yy} = 0. \tag{1}$$

The characteristics of (1) are real for $y \leq 0$ and consist of the two families

$$\frac{dy}{dx} = +(-y)^{-1/2} \tag{2a}$$

$$\frac{dy}{dx} = -(-y)^{-1/2} \tag{2b}$$

For positive values of y the *normal curves*, introduced by Tricomi, play an important role. These are the curves

$$(x - a)^2 + \frac{4}{9}y^3 = b^2. \tag{3}$$

Let $x = x(s)$, $y = y(s)$, $0 \leq s \leq s_0$, represent a simple rectifiable arc Γ in the half-plane $y > 0$ with end points on the x-axis at $A\ (-1, 0)$ and $B(1, 0)$. We suppose that $x(s)$, $y(s)$ have continuous third derivatives with respect to arc length s. Denote by Γ_0 the normal curve through A and B, *i.e.*, the curve given by $x^2 + 4y^3/9 = 1$. Let Γ contain Γ_0 in its interior except in some (arbitrarily small) neighborhood of A and B where Γ and Γ_0 coincide. Denote by Γ_1 the characteristic of (2b) issuing from A and by Γ_2 the characteristic of (2a) from B. The curves Γ, Γ_1, Γ_2 enclose a domain D_1 and Γ_0, Γ_1, Γ_2 a domain D_0. The part of D_1 above the x-axis we denote by D_1', the part below, D_1''. Let $\gamma : y = g(x)$ be an arc in D_1'' with the properties: (i) γ passes through the point A; (ii) there is a constant $M > 0$ such that $-M < g'(x) < 0$ for all x; (iii) in some (arbitrarily small) neighborhood of A, $g(x)$ coincides with Γ_1; (iv) γ intersects Γ_2 at some point C. Then Γ, γ and Γ_2 enclose a domain which we denote by D and the part below the x-axis by D''.

Let $f_1(s)$, $0 \leq s \leq s_0$, be a given function with a continuous second derivative and $f_2(s_1)$ a given function with a continuous fourth derivative defined on γ (where s_1 is arc length along γ).

The boundary value problem considered by Frankl consisted in determining a solution of (1), in D subject to the boundary conditions

$$u = f_1(s) \quad \text{on} \quad \Gamma, \qquad u = f_2(s) \quad \text{on} \quad \gamma. \tag{4}$$

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