MODULAR TRANSFORMATION OF CERTAIN SERIES

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1. **Introduction.** Modular functions can be expanded into infinite series by the methods developed by Hardy, Littlewood, Ramanujan, and Rademacher. See [4].

The converse problem of showing directly the behavior of these series under modular transformation was investigated by Rademacher [5]. In that paper he showed directly that the series representation of the function $J(\tau)$ is invariant under the modular transformations:

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1 \qquad (a, b, c, d, \text{ integers}). \tag{1.1}$$

It is the purpose of this paper to show how the methods of Rademacher [5] can be extended to include functions belonging to modular subgroups, and functions whose transformation equations include some rather complicated roots of unity.

We shall show directly from its series expansion that the function

$$\begin{gathered} f(x) = \prod_{n=1}^{\infty} (1 + x^n) = e^{-\pi i \tau/12} \frac{\eta(2\tau)}{\eta(\tau)} \\ x = e^{2\pi i \tau}, \qquad \eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \end{gathered} \tag{1.2}$$

belongs to the modular subgroup $\Gamma_0(2)$ consisting of those transformations in (1.1) for which $c \equiv 0 \pmod 2$. The proof will include a derivation of the transformation equation showing the behavior of $f(x)$ under the transformations of the group.

2. **Statement of the theorem.** For integers $h, k, (h, k) = 1$ define

$$\Omega_{h,k} = \left(\frac{2}{h}\right) \exp \pi i \left\{\frac{kh}{8} - \frac{1}{12}(2h - \bar{h} + h^2\bar{h})\left(\frac{k}{2} + \frac{1}{k}\right)\right\} \tag{2.1}$$

when k is even,

$$\Omega_{h,k} = \left(\frac{2}{k}\right) \exp \pi i \left\{\frac{1}{12}\left(k - \frac{1}{k}\right)\left(2h + \frac{\bar{h}}{2} + h^2\bar{h}\right)\right\} \tag{2.2}$$

when k is odd and $\bar{h}$ is even.

Received December 5, 1952. Presented to the International Congress of Mathematicians, September 4, 1950. This paper is based on the author's doctoral dissertation.