RELATIONS BETWEEN RIEMANNIAN AND HERMITIAN GEOMETRIES

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Introduction. Complex manifolds with an Hermitian or a Kählerian metric have become a topic of current interest in differential geometry. It is well-known that every complex manifold can be made Hermitian. However, a general differentiable manifold does not possess a complex structure, while the existence of a Kählerian metric imposes even more severe restrictions on the topological properties of the manifold. (For accounts of some of the recent works on the subject, we refer to papers [5], [6].) Following some questions raised to me by B. Eckmann, we study in this paper the properties of an analytic Riemannian manifold which locally behaves as an Hermitian or a Kählerian manifold. It turns out that the conditions for these properties can be expressed in a simple way in terms of the curvature tensor. We shall also show by examples that these properties are weaker than the corresponding global properties. On the other hand, we do not know whether any differentiable manifold can be given a locally Kählerian metric. In the four-dimensional case we shall prove that, if a further condition is satisfied, it does mean a restriction on the topological properties of the manifold.

By an Hermitian metric on a complex manifold is meant a positive definite Hermitian differential form which, in the local coordinates z^k, is given by

$$ds^2 = h_{jk}\, dz^j\, d\bar{z}^k \qquad (j, k = 1, \cdots, n), \tag{1}$$

where the functions h_{jk} satisfy the conditions

$$\bar{h}_{jk} = h_{kj}\ , \tag{2}$$

and their real and imaginary parts are real analytic in the arguments. (We agree, unless otherwise specified, that small Latin indices run from 1 to n, small Greek indices run from 1 to $2n$, and that repeated indices imply summation. Also a bar over a complex number denotes its complex conjugate.) The Hermitian metric is called Kählerian, if the corresponding exterior differential form is closed:

$$d(h_{jk}\, dz^j \wedge d\bar{z}^k) = 0. \tag{3}$$

Consider now an analytic Riemannian metric of $2n$ dimensions. We say that it is *locally Hermitian*, if in every neighborhood there exist complex-valued analytic functions z^k in the local coordinates x^α:

$$z^k = z^k(x^\alpha) \qquad (\alpha = 1, 2, \cdots, 2n) \tag{4}$$

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