MULTIVALENTLY STAR-LIKE FUNCTIONS

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1. **Introduction.** Let $S(p)$ denote the class of functions

$$f(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots , \tag{1.1}$$

regular and multivalently star-like with respect to the origin of order p in the unit circle $|z| < 1$ [8]. This means geometrically that, for a range $\rho < r < 1$, the image curve C_r of $|z| = r$, through the mapping $w = f(z)$, has the property that the vector joining the origin to the point $f(z)$ turns continuously through an angle $2p\pi$ in the anti-clockwise direction as z traverses the circle $|z| = r$ once in the same direction. Analytically, the functions $f(z)$ of (1.1) are characterized by the conditions

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad \int_0^{2\pi} \Re\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2p\pi, \tag{1.2}$$

for $z = re^{i\theta}$, $\rho < r < 1$. It is seen at once that $f(z)$ has exactly p zeros in $|z| < 1$.

For functions $f(z)$ with a power series (1.1) which are multivalent of order p (but not necessarily star-like) in $|z| < 1$ it was shown by Biernacki [1] that for $n > q$

$$|a_n| \leq A(p) \max \{|a_1|, \cdots, |a_q|\} n^{2p-1}, \tag{1.3}$$

when $f(z)$ has q zeros in $|z| < 1$. Goodman [3] has conjectured that perhaps (1.3) may be sharpened to be

$$|a_n| \leq \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)} |a_k| \tag{1.4}$$

for $n > p$. A great deal of evidence has piled up during the past four decades to indicate that (1.4) is true in the univalent case $p = 1$ ($|a_n| \leq n|a_1|$), although a proof has not been found except for several important sub-classes.

For the class $S(1)$, (1.4) is known to be true [6]. For the class $S(2)$, Goodman [4] has shown that (1.4) is correct for $n = 3$, provided all the coefficients a_n are *real*. In this case

$$|a_3| \leq 5|a_1| + 4|a_2|. \tag{1.5}$$

The method of proof of (1.5) made use of the approximating polygonal functions obtained by the Schwarz-Christoffel transformation. It was stated [4] that a similar proof gives the sharp inequality

$$|a_{p+1}| \leq (p-1)(2p+1)|a_{p-1}| + 2p|a_p|, \tag{1.6}$$

when $a_1 = a_2 = \cdots = a_{p-2} = 0$, and all the coefficients are real.

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