A CLASS OF EVERYWHERE BRANCHING SETS

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Introduction. Let P be a partially ordered set. For p an element in P, let $A(p) = \{x/x \geq p, x \varepsilon P\}$. One definition for P to be directed is the following: "A partially ordered set is directed if, for each pair of elements p and q in P, $A(p)$ is cofinal in $A(q)$ and $A(q)$ is cofinal in $A(p)$." It seems natural therefore to discuss a partially ordered set which has the following property: "For each pair of elements p and q in P, at least one of the two statements, (a) $A(p)$ is cofinal in $A(q)$, and (b) $A(q)$ is cofinal in $A(p)$, is false." Such a partially ordered set will be said to have "sufficiently many non-cofinal subsets." In Theorem 1 it is shown that if P is an everywhere branching ramified system, then P contains a cofinal subset S which has sufficiently many non-cofinal subsets. A subset Q of P shall be called "maximal residual" if (a) Q is a residual subset of P, and (b) Q is no proper cofinal subset of any residual subset of P. Let $F(P)$ denote the family of maximal residual subsets of P, partially ordered by the dual of set inclusion. $F(P)$ has sufficiently many non-cofinal subsets (Theorem 4). In Theorem 3 it is shown that if P and Q are any two everywhere branching, cofinally similar, partially ordered sets, then $F(P)$ is isomorphic to $F(Q)$.

1. **Two examples.** It shall be assumed that each partially ordered set P mentioned throughout this paper is non-empty and contains no maximal element.

Let M and N be two non-empty subsets of the partially ordered set P. M is said to be cofinal in N if, to each element p in N, there exists an element q in M such that $q \geq p$.

A partially ordered set P is said to be "everywhere branching" if, for each element p in P, there exist two elements, q and r, in P, such that $q \geq p$, $r \geq p$, and $A(q) \cap A(r) = \phi$ [1].

A useful characterization of a partially ordered set which has sufficiently many non-cofinal subsets is contained in the following easily proved lemma.

Lemma. *A partially order set P has sufficiently many non-cofinal subsets if and only if the elements of P have the following two properties:*

(a) *If p and q are any two elements of P for which $p > q$, then there exists an element r of P, $r > q$, such that $A(p) \cap A(r) = \phi$;*

(b) *If p and q are any two incomparable elements of P, then an element, r or s, of P can be found for which either $r > p$ and $A(r) \cap A(q) = \phi$, or $s > q$ and $A(p) \cap A(s) = \phi$.*

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