THE HARMONIC ANALYSIS OF BOUNDED FUNCTIONS

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1. **Introduction.** Recent advances in the theory of group algebras reveal that a serious gap exists in the harmonic analysis of functions bounded on the real line. In analytic form the problem left open is that of *spectral synthesis* [4], which may be stated in the following form. Let ϕ be a bounded measurable function on $(-\infty, \infty)$. By its *spectrum* $\sigma(\phi)$ we mean the set of real numbers λ with the following property: if h is a function in $L(-\infty, \infty)$ such that

$$h * \phi \equiv \int h(x - y)\phi(y)\, dy = 0$$

for all real values of x, then the Fourier transform of h

$$H(t) = \int e^{ixt} h(x)\, dx$$

vanishes at $t = \lambda$. (We omit limits from doubly infinite integrals.) The problem is this: is it true that any function h in $L(-\infty, \infty)$ whose Fourier transform vanishes on $\sigma(\phi)$ has the property $h * \phi \equiv 0$?

It follows from a result of Agmon and Mandelbrojt [1] that under the additional condition $xh(x) \in L$ the conclusion is correct. In his unpublished Harvard lectures (1949) Beurling showed that the weaker auxiliary condition $|x|^{1/2}h(x) \in L$ is sufficient. The main purpose of this paper is to give some improvements of these results (§11).

At the same time, in order to make available to analysts the results of a fragmentary literature, we present a brief account of the spectral theory of bounded functions. In place of the harmonic transform used in one form or another by previous writers, [1], [3], [7], we use Riemann's device of two integrations to obtain a spectral function. This makes it possible to give an exposition which does not invoke complex variable theory. As a consequence we obtain simple new real variable proofs of Wiener's Tauberian theorem (§3, §5) and Beurling's theorem on almost periodic functions (§10).

A full account of the problem from the algebraist's point of view, together with an exhaustive bibliography, has been given by Eberlein [8].

2. **An alternative definition of the spectrum.** Let $\phi(x)$ be a measurable function bounded on $(-\infty, \infty)$. Define a generalized Fourier transform by

$$F(t) = \int_{|x|\geq 1} \frac{e^{ixt}}{-x^2}\, \phi(x)\, dx + \int_{-1}^{1} \frac{e^{ixt} - 1 - ixt}{-x^2}\, \phi(x)\, dx. \tag{2.1}$$

This is simply Bochner's 2-transform [5]. $F(t)$ is continuous for all values of t.

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