BASIS THEOREMS FOR PARTIAL DIFFERENTIAL EQUATIONS

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Introduction. Many of the methods by which partial differential equations are solved require some form of superposition. Two facts emerge as characteristic of these methods. First, the superposition integral or series represents a linearly closed manifold of functions; that is, if f and g are representable by the given superposition process, so is $af + bg$ for any constants a and b. Second, the relevant differential equations are linear. It is our purpose to see in what sense these conditions are equivalent. Evidently the solutions of a linear homogeneous system form a linear manifold; is it true, conversely, that a linear manifold, which satisfies some partial differential equation, satisfies a linear homogeneous system? If so, is there a linear system such that the totality of its solutions reproduces the linear manifold exactly? Does continuity of the prescribed family ensure local existence of a continuous linear system with this as solution? Can one pass then to continuity in the large? Such are the questions with which the present discussion is concerned. Similar problems have been treated by Ritt in his well-known work on algebraic differential manifolds. Here, however, we assume nothing about the prescribed differential equation, except perhaps continuity, and the algebraic methods of Ritt seem inapplicable.

Definitions and notations. What is a partial differential equation? First, there must be a set of n independent variables $x, y, \cdots, w$, denoted here by the n-dimensional vector $\mathbf{X}$:

$$\mathbf{X} = (x, y, \cdots, w).$$

Next, one requires a dependent variable $U = U(\mathbf{X})$, and certain of its partial derivatives:

$$\mathbf{V} = (U, U_x, U_{xyy}, U_{yzw}, \cdots). \tag{1}$$

The dimension of $\mathbf{V}$ is taken as a finite number, m. Finally, a *partial differential equation* is a relation of the form

$$F(\mathbf{X}, \mathbf{V}) = 0, \tag{2}$$

where F is a function of its $m + n$ arguments, well-defined over the range in which the equation is supposed to hold. For simplicity we assume F, $\mathbf{X}$, and $\mathbf{V}$ real.

We say that U is a *solution* of (2) in a set S of $\mathbf{X}$-values if the vector $\mathbf{V}$ obtained by differentiating U satisfies (2) at each point of S. It may be mentioned

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