REDUCTION OF DIFFERENTIAL SYSTEMS TO FIRST ORDER

By Robert Taylor Herbst

1. **Introduction.** An ordinary differential equation of order n in one unknown is equivalent to a system of n equations of the first order in n unknowns. This result (attributed to d'Alembert by Ince [2; 14]) is easily rendered precise, its proof is immediate and it is advantageously applied to obtain information about the equation of order n from facts known about systems of the first order [2; 73]. On the other hand, the situation for general systems in either ordinary or partial derivatives is not so simple.

The equivalence of a system of partial differential equations to a system of the first order is frequently mentioned as well known. This statement must be construed as referring to a formal substitution which usually is non-informative because the system of the first order fails to be passive (self-contained) or normal (in a form for which existence has been proved). The difficulties involved were recognized and overcome by Riquier [3], who gave a reduction to a passive, normal system of the first order. In proving the Cauchy-Kowalevsky existence theorem Goursat [1; 635] incidentally gave a direct reduction for the relatively simple system involved.

The present paper gives for certain passive systems a reduction to a passive system of the first order. It is applicable to systems defined by functions continuously differentiable to an order for which a bound can be specified. It places the burden of existence proofs for such systems upon the discussion of like systems of the first order. It is applicable in particular to orthonomic systems to which Riquier's process is restricted and seems more direct than Riquier's.

2. **Definition of the systems.** A general exposition of the basic ideas is given in [4]. Here is given a brief summary of those aspects of the theory essential for the reduction. At the same time it is found advantageous to modify the notation and nomenclature of [4] slightly.

Let $u_1, \cdots, u_r$ denote indeterminates to be called *unknowns.*

Let $x_1, \cdots, x_n$ denote indeterminates to be called *variables.*

The expression $u_i x_1^{j_1} \cdots x_n^{j_n}$ is the *derivative* of u_i with respect to the *monomial* $x_1^{j_1} \cdots x_n^{j_n}$, differentiation being formal multiplication by a variable.

If u_i, x_j are assigned arbitrary fixed values from the real field, the derivatives D assume real values which at least partially order them. If the values assigned satisfy

$$(2.1) \qquad 0 < u_i, \qquad 1 < x_j,$$

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