ANALYTICITY, AND THE MAXIMUM MODULUS PRINCIPLE

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It is well known that analytic functions of a complex variable satisfy the maximum modulus principle: if f is analytic and single-valued in a domain (*i.e.*, a connected open set) D, then $|f(z)|$ cannot attain a proper maximum at any point of D. One may now ask to what extent the maximum modulus principle characterizes the class of analytic functions, and it is this question which will occupy us in the present paper.

We cannot, of course, expect any results in this direction for single functions; but the situation becomes interesting if we consider algebras of functions. Let us recall that $\mathfrak{A}$ is an algebra if $f + g \in \mathfrak{A}$, $fg \in \mathfrak{A}$, and $cf \in \mathfrak{A}$, whenever $f \in \mathfrak{A}$, $g \in \mathfrak{A}$, and c is a complex number.

Let B be the boundary of a compact set K in the plane. We say that an algebra $\mathfrak{A}$ of complex-valued functions continuous on K is a *maximum modulus algebra* on K if for every $f \in \mathfrak{A}$ there is a point $z_0 \in B$ such that

$$|f(z)| \leq |f(z_0)| \qquad (z \in K).$$

We say that $\mathfrak{A}$ is a *local maximum modulus algebra* in the domain D if for every $z \in D$ there is a sequence of Jordan domains U_n whose diameter tends to zero as $n \to \infty$, such that $z \in U_n$, and such that $\mathfrak{A}$ is a maximum modulus algebra on the closure of U_n (by a Jordan domain we mean, as usual, the interior of a simple closed curve).

Our principal results follow:

Theorem 1. *Let $\mathfrak{A}$ be a maximum modulus algebra on the closure K of a Jordan domain D. If $\mathfrak{A}$ contains a function ψ which is schlicht* (*i.e., one-to-one*) *on K, then every member of $\mathfrak{A}$ is an analytic function of ψ. If, in addition, $\mathfrak{A}$ contains a non-constant function ϕ which is analytic in D, then every member of $\mathfrak{A}$ is analytic in D.*

The first part of the conclusion may be stated more explicitly in the following manner: for every $f \in \mathfrak{A}$ there is a function f^* continuous on $\psi(K)$, analytic in $\psi(D)$, such that $f(z) = f^*(\psi(z))$ for $z \in K$.

Theorem 2. *If $\mathfrak{A}$ is a local maximum modulus algebra in a domain D, and if $\mathfrak{A}$ contains a function ϕ which is analytic and not constant in D, then every member of $\mathfrak{A}$ is analytic in D.*

Our proof of Theorem 4 which in a certain sense extends Theorem 1 to multiply connected domains will depend on

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