ASYMPTOTIC PROPERTIES OF FUNCTIONS OF EXPONENTIAL TYPE

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1. **Introduction.** Suppose that $f(z)$ is regular and of exponential type in the half plane $y \geq 0$, and bounded on the real axis. Let

$$h(\theta) = \limsup_{r \to \infty} r^{-1} \log | f(re^{i\theta}) |;$$

simple arguments [34; vol. 2, p. 36, problem 202], [12] show that not only is $h(\theta) = c \sin \theta$ with some constant c, but even that $| f(z) | \leq Me^{cy}$. However, much more than this can be said, as the following theorem of Ahlfors and Heins [3] (see also [19]) shows.

THEOREM 1. *If $f(z)$ is regular and of exponential type in $y \geq 0$, and $| f(x) | \leq 1$, then $\lim_{r \to \infty} r^{-1} \log | f(re^{i\theta}) | = c \sin \theta$ for $0 < \theta < \pi$ with the exception at most of a set of outer capacity zero; uniformly in any closed interior angle if r is excluded from a set of finite logarithmic length; and without exception in any interior angle which contains no zeros of $f(z)$.*

(The original theorem deals more generally with a subharmonic function instead of with $\log | f(z) |$.)

Let us express the conclusion of Theorem 1 by saying that $r^{-1} \log | f(z) |$ tends effectively to $c \sin \theta$. By the statement that $r^{-1} \log | f(z) |$ is effectively bounded we shall mean the same thing with boundedness replacing approach to a limit.

It is well known that the behavior of $f(z)$ inside the upper half plane is strongly affected by its behavior on the real axis. It is therefore of interest to see how much the condition that $| f(x) |$ is bounded can be weakened without destroying the conclusion. We shall establish the following results.

THEOREM 2. *If $f(z)$ is regular and of exponential type in $y \geq 0$ and*

$$(\mathrm{L}^+) \qquad \int_{-\infty}^{\infty} \frac{\log^+ | f(x) |}{1 + x^2} dx < \infty,$$

the conclusion of Theorem 1 holds.

THEOREM 3. *If $f(z)$ is regular and of exponential type in $y \geq 0$ and*

$$(\mathrm{I}_\pm) \qquad \int_1^{\to \infty} x^{-2} \log | f(\pm x) | \, dx \qquad \textit{exist},$$

the conclusion of Theorem 1 holds.

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