SOME THEOREMS ON KUMMER'S CONGRUENCES

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1. **Introduction.** Let p be a fixed prime and let $\{a_m\}$ be a sequence of rational numbers that are integral (mod p); somewhat more generally we may suppose that the a_m are integral p-adic numbers. Let

$$(1.1) \qquad \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s} a_{m+s(p-1)} a_p^{r-s} \equiv 0 \pmod{p^r}$$

for all $m \geq r \geq 1$. We shall call (1.1) Kummer's congruence for $\{a_m\}$. For example (1.1) holds for $p > 2$, $a_m = E_m$, the Euler number in the even suffix notation. It is sometimes convenient to assume a little less, namely that $p - 1 \nmid m$, in which case we take $m \geq r + 1$; this is the case when $a_m = B_m/m$, where B_m is the Bernoulli number in the even suffix notation (see for example [6; Chapter 14]). For simplicity we shall usually assume that (1.1) holds for all $m \geq r \geq 1$.

In this note we first prove the following two theorems.

Theorem 1. *If $\{a_m\}$ satisfies (1.1) and $\{b_m\}$ satisfies a like congruence then the same is true for $\{c_m\} = \{a_m b_m\}$.*

Theorem 2. *Let $c_m^{(k)} = m^k a_m$, $k \geq 1$. If $\{a_m\}$ satisfies (1.1) then*

$$(1.2) \qquad \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s} c^{(k)}_{m+sp(p-1)} a_p^{r-s} \equiv 0 \pmod{p^r}.$$

Extensions of these theorems will be found in Theorems 1′, 3, 4 below. A number of applications are also given.

Finally we consider Kummer's congruences for the sequence $\{c_m\}$, where

$$(1.3) \qquad c_m = \sum_{s=0}^{m} \binom{m}{s} a_s b_{m-s}.$$

Put $f(x) = \sum_1^\infty a_m x^m/m!$, $g(x) = \sum_1^\infty b_m x^m/m!$. If we assume that $a_p \equiv b_p \pmod{p}$, and

$$(1.4) \qquad (D^p - a_p D) f(x) = p \sum_{0}^{\infty} A_m f^m(x),$$

where the A_m are integral (mod p), and a like formula for $g(x)$, then we can prove that c_m, defined by (1.3), satisfies

$$(1.5) \qquad \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s} c_{m+s(p-1)} a_p^{r-s} \equiv 0 \pmod{p^r}.$$

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