PASCAL'S THEOREM IN SPACE

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1. **Introduction.** a. Pascal's theorem may be stated as follows. If the three pairs of sides of a triangle (t) passing through the three vertices are cut by three transversals, the six points of intersection lie on a conic, if, and only if, the three transversals cut the respective third sides of (t) in three collinear points.

b. The three transversals considered (§1a) may have a point in common, or may form a triangle (t'). In the first case we have the proposition: If three collinear points, taken on the sides of a triangle (t), are projected from a point, the three projecting lines cut the remaining sides of (t) in six points lying on a conic.

c. If the three transversals form a triangle (t'), we have: If two triangles are perspective, the three sides of one meet the pairs of noncorresponding sides of the other in six points lying on a conic [2: 123, article 155].

d. Brianchon's theorem may be stated as follows. If the three pairs of vertices lying on the three sides of a triangle (t) are projected from three points, the six projecting lines are tangent to a conic, if, and only if, the three lines joining the centers of projection to the respective vertices of (t) are concurrent.

2. **Pascal's theorem in space.** a. The four triads of edges of a tetrahedron $(T) = ABCD$ passing through the vertices A, B, C, D are cut, respectively, by four planes α, β, γ, δ in four triads of points. How shall the four transversal planes be chosen in order that the twelve points of intersection shall lie on a quadric surface?

b. Let the plane α cut the edges AC, AB of the face ABC of (T) in the points Y_a, Z_a. The point $X = (BC, Y_aZ_a)$ lies in the three planes ABC, DBC, and α, hence X is the trace of the line $p = (\alpha, DBC)$ in the plane ABC, or, more precisely, on the edge BC.

Similarly if the planes β, γ cut the pairs of edges BC, BA; CB, CA, respectively, in the pairs of points X_b, Z_b; X_c, Y_c, the points $Y = CA, Z_bX_b)$, $Z = (AB, X_cY_c)$ are the traces of the lines $q = (\beta, DCA)$, $r = (\gamma, DAB)$ on the edges CA, AB of the face ABC.

In order that the six points

(e) $$Y_a, Z_a; Z_b, X_b; X_c, Y_c$$

in which the planes α, β, γ cut the edges of the face ABC shall lie on a conic (q_d) it is thus necessary that the three points X, Y, Z shall be collinear (§1a).

Similarly for the other faces of (T). Thus in order that the sixes of points

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