METHODS OF CONSTRUCTING CERTAIN STOCHASTIC MATRICES

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1. **Introduction.** The problem of determining necessary and sufficient conditions for the $n + 1$ real numbers $1, \lambda_1, \cdots, \lambda_n$, where $|\lambda_i| \leq 1$ for $i = 1, 2, \cdots, n$, to be a possible set of characteristic roots of a stochastic matrix of order $n + 1$ has not yet been fully solved. Suleĭmanova [3] describes a geometrical technique for discussing such conditions, and claims that, using this technique, it is possible to prove that the real numbers $1, \lambda_1, \cdots, \lambda_n$, where $|\lambda_i| < 1$ for $i = 1, 2, \cdots, n$, are characteristic roots of a positive, simple stochastic matrix of order $n + 1$ provided that the sum of the moduli of the negative numbers of the set is less than unity; other allied results for general (*i.e.* not necessarily strictly positive) stochastic matrices are stated in [3]. In a recent paper [2] I applied Suleĭmanova's technique to the cases $n = 2, 3$ and proved that the numbers $1, \lambda_1, \cdots, \lambda_n$, where $|\lambda_i| < 1$ for $i = 1, 2, \cdots, n$, are characteristic roots of a positive, simple, stochastic matrix of order $n + 1$ if and only if $1 + \sum_{i=1}^{n} \lambda_i$ is positive. (The full discussion of the case $n = 3$ was not included on account of its length.) The method which I used in [2] in each case yielded a matrix for actually performing the transformation of the given real diagonal matrix diag $(1, \lambda_1, \cdots, \lambda_n)$, $n = 2, 3$, into a positive stochastic matrix. For example, the matrix P diag $(1\lambda_1\lambda_2)P^{-1}$, where $1 > \lambda_1 \geq \lambda_2 \geq 0$ and

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix},$$

is a positive stochastic matrix. As a footnote to [2] I shall, in Theorem 1 of §2 below, generalize this particular example. For the most part however in this paper I shall reject the geometrical technique of Suleĭmanova's, and, applying a theorem (Theorem 2 in §3) essentially due to A. Brauer [1], I shall obtain two theorems (Theorems 3, 4 in §4) giving certain *sufficient* conditions for the set of real numbers $1, \lambda_1, \cdots, \lambda_n$, where $|\lambda_i| \leq 1$ for $i = 1, 2, \cdots, n$, to be a possible set of characteristic roots of a stochastic matrix of order $n + 1$.

2. **A generalization of a result proved in** [2]. It follows very simply from the theory to be described in §§4, 5 that the set of $n + 1$ real numbers $1, \lambda_1, \cdots, \lambda_n$,

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