LEAST DETERMINANTS OF INTEGRAL QUADRATIC FORMS

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O'Connor and Pall [1] have determined the least possible determinant for positive definite integral quadratic forms. The signature of such a form in s variables is s. If the least determinant is $\delta(s, s)$, then

$$\delta(s, s) = 1/2^s \ (s \equiv 0), \qquad 2/2^s \ (s \equiv \pm 1),$$
$$3/2^s \ (s \equiv \pm 2), \qquad 4/2^s \ (s \equiv \pm 3, 4),$$

the congruences being modulo 8. Representative s-ary forms f_s are given in each case.

In another connection I have had to find the numerically least determinants for indefinite integral quadratic forms. The results naturally depend on those of O'Connor and Pall and also on a theorem of Meyer (see Dickson [2]) which states that an integral indefinite form in 5 or more variables must represent zero properly, *i.e.* for integers with greatest common divisor unity.

2. Let g_m be an integral quadratic form in m variables with signature $\pm s$ and rank m, where $s \geq 0$ so that

$$m = s + 2r$$

for integral $r \geq 0$. If $r = 0$, the form will be definite. Let $\Delta(g)$ denote the determinant of g_m and let $\delta(m, s)$ denote the minimum of $| \Delta(g) |$. We then have the following theorem.

Theorem. *$\delta(m, s)$ has the values*

$$1/2^m \ (s \equiv 0), \qquad 2/2^m \ (s \equiv \pm 1), \qquad 3/2^m \ (s \equiv \pm 2),$$
$$4/2^m \ (s \equiv \pm 3, 4 \pmod 8).$$

Forms for which equality occurs are

$$g_m = x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r} \pm f_s(y_1, \cdots, y_s),$$

where $f_s = 0$ if $s = 0$, and f_s $(s \geq 1)$ is the form defined by O'Connor and Pall:

$$f_1 = y_1^2, \qquad f_2 = y_1^2 + y_1y_2 + y_2^2, \qquad f_3 = f_2 + y_2y_3 + y_3^2,$$
$$f_4 = f_3 + 2y_3y_4 + 2y_4^2$$
$$f_5 = \sum_{i=1}^{3} (y_i^2 + y_iy_{i+1}) + y_4^2 + 2y_4y_5 + 2y_5^2,$$
$$f_j = \sum_{i=1}^{j-2} (y_i^2 + y_iy_{i+1}) + y_{j-1}^2 + 3y_{j-1}y_j + 4y_j^2 \qquad (j = 6, 7, 8),$$

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