THE ASYMPTOTIC BEHAVIOUR OF POWERS OF MATRICES. II.

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This note is an addendum to our paper [3]. We will extend the results obtained in II of [3] by introducing more general norms, and from this we derive further sequences converging towards ω_A. The enumeration of the equations, theorems and sections will be continued from [3].

III. Generalizations

1. To any column-vector $x = (x_1, \cdots, x_n)$ of the n-dimensional complex Euclidean space let a number $\phi(x)$, called the *norm* of x, be assigned, satisfying the following three conditions:

(i) $\phi(x) > 0$ except for the null vector $x = 0$, for which $\phi(0) = 0$,
(ii) $\phi(\lambda x) = |\lambda| \phi(x)$ for any complex scalar λ,
(iii) $\phi(x + y) \leq \phi(x) + \phi(y)$.

Furthermore, suppose that $\phi(x)$ is bounded over the set of vectors with Euclidean length $|x| = 1$,

$$\text{(iv)} \qquad \phi(x) \leq C \qquad (|x| = 1).$$

Let a function of the vector x, $\psi(x)$, satisfy (i) and (ii) and be bounded from below by a *positive* constant for all vectors x with $|x| = 1$,

$$\text{(v)} \qquad \psi(x) \geq c > 0 \qquad (|x| = 1).$$

Then for an $n \times n$ matrix $A = (a_{\nu\mu})$ the ratio $\phi(Ax)/\psi(x)$ remains bounded over the set of all vectors $x \neq 0$; we may therefore define its least upper bound

$$\Omega_{\phi,\psi}(A) \equiv \sup_{x \neq 0} \frac{\phi(Ax)}{\psi(x)} \tag{28}$$

as the (upper) *norm of A induced by ϕ and ψ*. (Compare for this definition A. Ostrowski [5].) $\Omega_{\phi,\psi}(A)$ is a special case of the most general norm $\Omega(A)$ defined by the three properties:

(vi) $\Omega(A) > 0$ except when $A = 0$, in which case $\Omega(0) = 0$,
(vii) $\Omega(\lambda A) = |\lambda| \Omega(A)$ for any complex scalar λ,
(viii) $\Omega(A + B) \leq \Omega(A) + \Omega(B)$, A, B being $n \times n$ matrices.

If in particular we take $\psi(x) = \phi(x)$ assuming of course that ϕ satisfies (v), $\Omega_\phi \equiv \Omega_{\phi,\phi}$ also satisfies

$$\text{(ix)} \qquad \Omega_\phi(AB) \leq \Omega_\phi(A)\Omega_\phi(B) \qquad (\Omega_\phi \equiv \Omega_{\phi,\phi}),$$

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