THE DISTRIBUTION MODULO 1 OF TRIGONOMETRIC SEQUENCES

BY W. J. LEVEQUE

1. In an earlier paper [4] I proved a general theorem which, when suitably specialized, has the following easy consequence:

If $X = (x_1, \cdots, x_r)$ *is an r-dimensional continuous variable, R is a set having positive r-dimensional Lebesgue measure, and* $\{f_n(X)\}$ *is a sequence of real-valued functions such that*

$$\left| \int_R \exp (i(f_n(X) - f_m(X)))\, dX \right| \leq \frac{C}{\max (1, |n - m|^\epsilon)} \qquad (C, \epsilon > 0), \tag{1}$$

then $\{f_n(X)\}$ *is uniformly distributed* (mod 1) *for almost all* $X \varepsilon R$. *Moreover, in the three cases* $0 < \epsilon < 1$, $\epsilon = 1$, $\epsilon > 1$ *the respective bounds*

$$\sum_{n=1}^{N} \exp (2\pi i \beta f_n(X)) = \begin{cases} O(N^{1-\delta}) & (\delta < \epsilon/2) \\ O(N^{\frac{1}{2}} \log^{5/2+\theta} N) & (\theta > 0) \\ O(N^{\frac{1}{2}} \log^{3/2+\theta} N) & (\theta > 0) \end{cases} \tag{2}$$

hold for almost all $X \varepsilon R$. *Here* β *is an arbitrary fixed integer different from zero.*

In [4] this was applied only to cases in which (1) can be trivially verified by integration by parts. We consider now some cases in which the assumptions made earlier, about the monotonicity and the size of certain derivatives of $f_n(X) - f_m(X)$, do not hold.

Suppose that $z = re^{ix}$ is a fixed complex number with $r > 1$, and put $z^n = x_n + iy_n$, so that $x_n = r^n \cos nx$ and $y_n = r^n \sin nx$. If $\langle u \rangle$ denotes the fractional part of the real number u (that is, $u = [u] + \langle u \rangle$), then the points $(\langle x_n \rangle, \langle y_n \rangle)$ all lie in the unit square $0 \leq u_1 < 1$, $0 \leq u_2 < 1$; otherwise stated, the points $\langle x_n \rangle + i\langle y_n \rangle = \langle z^n \rangle$ all lie in the unit square $0 \leq \operatorname{Re} z < 1$, $0 \leq \operatorname{Im} z < 1$. According to van der Corput's generalization of Weyl's criterion (compare [1; 92]), a necessary and sufficient condition that they be uniformly distributed over the square is that, for every pair of integers β_1, β_2 not both zero,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \exp (2\pi i(\beta_1 x_n + \beta_2 y_n)) = 0.$$

We have

$$\begin{aligned} \beta_1 x_n + \beta_2 y_n &= r^n(\beta_1 \cos nx + \beta_2 \sin nx) \\ &= (\beta_1^2 + \beta_2^2)^{\frac{1}{2}} r^n \cos (nx - \alpha), \end{aligned}$$

where

$$\cos \alpha = \frac{\beta_1}{(\beta_1^2 + \beta_2^2)^{\frac{1}{2}}}, \qquad \sin \alpha = \frac{\beta_2}{(\beta_1^2 + \beta_2^2)^{\frac{1}{2}}}.$$

Received October 10, 1952.