

# COMPLETENESS THEOREMS FOR SETS OF DIFFERENTIAL OPERATORS

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1. **Introduction.** In the present paper, we shall concern ourselves with certain aspects of the following general problem. Let  $L_n$  ( $n = 0, 1, \dots$ ) be a set of linear differential operators of infinite order:

$$(1.1) \quad L_n(f) = \sum_{k=n}^{\infty} a_{nk} f^{(k)}(0)/k! \quad (a_{nn} = 1)$$

applicable to the members  $f$  of a class  $A$  of functions regular in a neighborhood of the origin. Can we distinguish a subclass  $S$  of  $A$  such that the conditions

$$(1.2) \quad L_n(f) = 0 \quad (n = 0, 1, \dots; f \in S)$$

imply  $f \equiv 0$ ? The functionals (1.1) with which we shall be dealing have representations as Cauchy integrals, and we may say alternatively that we shall be concerned with the determination of classes  $S$  for which the set  $\{L_n\}$  is complete. A set  $S$  with this property has also been called a uniqueness class for the set of functionals  $\{L_n\}$ .

There is an extensive literature associated with various special cases of this problem, and the reader is referred to Boas [2], Buck [5], Gontcharoff [11], where many bibliographical references will be found. The most precise results in this field occur when uniqueness classes are sought for specific sets of functionals, and, generally speaking, the most effective method of handling a specific set  $\{L_n\}$  is through the investigation of a related interpolation series

$$(1.3) \quad f(z) \sim \sum_{n=0}^{\infty} L_n(f) p_n(z); \quad L_m(p_n) = \delta_{mn}.$$

For certain general results, see Davis [6; 152-155] where the method of infinite systems of linear equations was employed. The present paper uses the same techniques of infinite systems of linear equations to arrive at necessary and sufficient conditions that various classes of interest be uniqueness classes for a preassigned set of linear functionals. (If it is desired to work with a Hilbert space of analytic functions, it is useful to orthonormalize the functionals  $L_n$ . Compare Walsh-Davis [17].) These criteria are then applied to the study of the specific sets

$$(1.4) \quad L_n(f) = f^{(n)}(a_n) \quad (n = 0, 1, \dots)$$

$$(1.5a) \quad L_{2n}(f) = f^{(2n)}(a_{2n}) \quad (n = 0, 1, \dots).$$

$$(1.5b) \quad L_{2n+1}(f) = f^{(2n)}(a_{2n+1})$$

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