CONGRUENCES FOR EULERIAN NUMBERS

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1. **Introduction.** The numbers of the title, in the first instance, are associated with the following exponential generating function given by Euler [3; 487–491]

$$\frac{1-x}{e^t - x} = \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!} \qquad (x \neq 1) \tag{1.1}$$

so that $H_k = H_k(x)$ is a rational function of x with integral coefficients. This is equivalent to

$$(H + 1)^k = xH_k + (1 - x)\delta_{k0} , \tag{1.2}$$

where δ_{k0} is a Kronecker delta and the left-hand side is symbolic: after expansion H^k is replaced by H_k .

Then, if

$$(x - 1)^k H_k(x) = A_k(x) = \sum_{s=1}^{k} A_{ks} x^{s-1} \qquad (k > 0), \tag{1.3}$$

the numbers A_{ks} are Eulerian numbers. They have a long and varied history, some of which is given in [4]; recent appearances are in [1] and [7].

Alternatively, using a relation due to Worpitsky [9], they may be defined by

$$x^k = \sum_{s=1}^{k} A_{ks} \binom{x + s - 1}{k} \qquad (k > 0), \tag{1.4}$$

and this may be generalized to (Shanks, [8], see also [2])

$$\binom{x}{i}^k = \sum_{s=1}^{ik-i+1} A_{ks}^{(i)} \binom{x + s - 1}{ik} \qquad (k > 0), \tag{1.5}$$

which reduces to (1.4) for $i = 1$ with $A_{ks} = A_{ks}^{(1)}$.

This is particularly interesting in view of the combinatorial interpretation by means of Simon Newcomb's problem (MacMahon [6], and [5]). This is associated with a game of patience played with a deck of cards of arbitrary specification, with cards dealt into a single pack so long as they are not falling and a new pack started for every fall; the problem is to determine the number of ways of dealing s packs. When the deck is of cards numbered 1 to k, the answer is given by numbers A_{ks} . As will appear, when the deck has k kinds of cards and i of each kind, *i.e.* has specification i^k, the answer is given by numbers, $A_{ks}^{(i)}$ of (1.5). Moreover (1.5) has a natural generalization to the numbers associated with an arbitrary deck, which satisfy congruences similar to those appearing below but we limit consideration, for brevity, to the numbers of (1.5).

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