PROPERTIES AND FACTORIZATIONS OF MATRICES DEFINED BY THE OPERATION OF PSEUDO-TRANSPOSITION

By Leo Katz and Ingram Olkin

1. **Introduction.** A matrix $C: n \times n$ (*i.e.*, of n rows and n columns) is (p,q) pseudo-orthogonal if it satisfies the relation

$$CJC' = J, \tag{1.1}$$

where C' is the transpose of C, $J = I_p \dot{+} (-I_q)$, $\dot{+}$ is the direct sum, I_p is the identity matrix of order p, and $p + q = n$. This implies the invariance of the quadratic form $x'Jx$, $x: n \times 1$ under a pseudo-orthogonal transformation. In this sense, a pseudo-orthogonal transformation is a rotation in a pseudo-Euclidean space of p and q dimensions. Throughout this paper, we shall consider p and q as fixed.

Many writers have investigated properties of pseudo-orthogonal matrices; in particular, Lee [1] and Hsu [2] have obtained factorizations of such matrices. Lorentz matrices and symplectic matrices (after a permutation on rows and columns) are examples of p-orthogonal matrices, the former being a special case with $p = 1$, $q = 3$.

By defining the operation of pseudo-transposition, we obtain unified definitions of pseudo-symmetric, pseudo-skew, and pseudo-orthogonal matrices (henceforth denoted by the prefix p-, *e.g.*, p-orthogonal), which are analogous to the definitions using ordinary transposition. Also, certain analogs of theorems involving transposition hold for p-transposition. We obtain, in Theorem 4.2, a new factorization of a p-orthogonal matrix in terms of a p-skew matrix, and in Theorem 5.2, the analog of the Toeplitz factorization (see [3; 80]). The matrices considered in this paper are real.

2. **Definitions.** Postmultiplication of both sides of (1.1) by J gives $C\ (JC'J) = I$, which is strongly reminiscent of the form $CC' = I$ for orthogonal matrices and suggests the Fundamental Operation: $C^0 = JC'J$ is the p-transpose of C. If

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix},$$

where $X: n \times n$, $X_1: p \times p$, $X_4: q \times q$, $p + q = n$, then

$$X^0 = \begin{pmatrix} X_1' & -X_3' \\ -X_2' & X_4' \end{pmatrix}. \tag{2.1}$$

Received August 11, 1952. This paper was prepared under the sponsorship of the Office of Naval Research.