DIFFERENTIAL OPERATORS OF INFINITE ORDER

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Introduction. The present paper is concerned with the theory of differential operators of infinite order which are of the form

$$G(D_z) = \sum_{k=0}^{\infty} g_k D_z^k ,$$

where $G(w)$ is an entire function of w and D_z is a linear differential operator of finite order s having analytic coefficients.

Several authors have treated the case in which $D_z = d/dz$. It is basic for the theory of Dirichlet series, that is, for series expansions in terms of the characteristic functions of the operator d/dz. We mention only the papers of H. Muggli [7], J. F. Ritt [9], and G. Valiron [12] which are closely related to our problem.

The case of an arbitrary linear first order operator

$$D_z = P_0(z)\, d/dz + P_1(z)$$

has been studied from the same point of view by Mary K. Peabody [8]. Starting with the theory of Hermite-Weber series, E. Hille [4, 5] considered the case in which

$$D_z = z^2 - d^2/dz^2,$$

that is, the differential operator of which the Hermite-Weber orthogonal functions form a set of characteristic functions.

In the present paper we examine the case of an arbitrary linear second order operator

$$D_z = P_0(z)\, d^2/dz^2 + P_1(z)\, d/dz + P_2(z) \tag{1}$$

and indicate the extension to $s > 2$. Our results are much more precise for $s = 2$. For this reason as well as for the increasing complexity of the formal algebra for larger values of s, we lay the emphasis on the case $s = 2$.

A basic property of the operator $G(D_z)$ is that it preserves holomorphism provided $G(w)$ is an entire function of sufficiently low order. The critical limit for the order of $G(w)$ is the reciprocal of the order of D_z . Thus, if $G(w)$ is an entire function of order $\sigma \leq 1/s$ and of minimal type if $\sigma = 1/s$, if $f(z)$ and the coefficients of D_z are holomorphic in a domain Δ of the complex plane, then $G(D_z)f(z)$ exists as a holomorphic function of z in Δ. In a certain sense this

Received August 9, 1952; presented to the American Mathematical Society, August 24, 1952. This paper was a dissertation presented for the degree of Doctor of Philosophy in Yale University. The author wishes to thank Professor Einar Hille for suggesting this problem and for his help in preparing this paper.