MODULAR CRITERIA ON RIEMANN SURFACES

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Introduction. In this paper, we establish criteria for the nonexistence of Green's functions and single-valued analytic functions with a finite Dirichlet integral on a given arbitrary Riemann surface.

In §1, it is proved that the surface is of parabolic type if and only if there exists an exhaustion with a divergent modular product. Existence of exhaustions with a convergent modular product is considered in §2. In §3, we derive connections of the above criterion with type criteria of Ahlfors [1], Laasonen [5] and Nevanlinna [6]. §4 contains a simplified proof of a modular criterion, introduced by the author in [8], for an AD-removable boundary.

The paper is a detailed exposition of results outlined in the preliminary notes [9], [11]. The sufficient type criterion introduced in [11] (Theorem 1 below), was proved by Noshiro [7] and Kuroda [3], [4] to be necessary as well. Heins [2] showed that the criterion is sufficient to guarantee even that the boundary be of harmonic dimension one.

1. **Modular criterion for the parabolic type.** Let R be an arbitrary open Riemann surface and $\{R_n\}$ an exhaustion of R, each R_n being bounded by a finite set β_n of closed analytic Jordan curves with $\beta_n \cap \beta_{n+1} = 0$. The difference $R_n - R_{n-1}$ consists of a finite number of subregions E_{ni}. Let s_n be the harmonic function in $R_n - R_{n-1}$ with $s_n = 0$ on β_{n-1}, $s_n = \log \sigma_n$ on β_n where σ_n (>1) is a constant such that

$$\int_{\beta_n} d\bar{s}_n = 2\pi. \tag{1}$$

Here, as everywhere in this paper, a barred letter stands for the conjugate harmonic function. The corresponding analytic function will be noted by a capital letter, $S_n = s_n + i\bar{s}_n$. If we cut each E_{ni} along certain level lines $\bar{s}_n =$ a constant so as to form a planar domain, and select the arbitrary constants of $\bar{s}_n$ in the E_{ni} properly, the function

$$F_n = e^{S_n} \tag{2}$$

maps $R_n - R_{n-1}$ onto a circular annulus with radii 1 and σ_n, cut along radial slits. We say that

$$\sigma_n = \text{modulus of } R_n - R_{n-1} \tag{3}$$

in the present exhaustion.

By definition, a Riemann surface R is of parabolic type if there are no Green's

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