PRODUCTS OF NORMAL OPERATORS

By Irving Kaplansky

1. **Introduction.** In [5] Wiegmann proved the following interesting theorem: *if A and B are matrices such that A, B, and AB are all normal, then BA is also normal.* In [6] he extended this to completely continuous operators.

In the present note we shall look into the validity of this result for general operators on a Hilbert space. We hasten to inform the reader of the fact (surprising to the author) that the result may be false; an example is given in §5. On the positive side of the ledger we contribute the following: a reduction of the problem, a trace argument, and a generalization of the completely continuous case.

2. **A reduction.** The following theorem accomplishes a reduction of the problem from one of the fourth degree (the normality of BA) to one of the third degree.

Theorem 1. *Let A and B be operators on Hilbert space such that A and AB are normal. Then the following statements are equivalent:* (1) *B commutes with A^*A,* (2) *BA is normal.*

Proof. (1) $\rightarrow$ (2). Form the polar decomposition $A = UR$. Since A is normal, U is unitary, and U and R commute. Also B commutes with R, the positive square root of A^*A. We have

$$U^*ABU = U^*URBU = BRU = BUR = BA.$$

Thus BA is unitarily equivalent to a normal operator and so is itself normal.

(2) $\rightarrow$ (1). The theorem of Fuglede [2], as generalized by Putnam [3], states the following: *if P and Q are normal and $PA = AQ$, then also $P^*A = AQ^*$.* We apply this with $P = AB$, $Q = BA$. The conclusion is $B^*A^*A = AA^*B^*$. In view of the normality of A, this says that B^* commutes with A^*A, and hence so does B.

3. **A trace argument.** With the aid of Theorem 1 we are able to give a trace argument for the normality of BA. It seems that such a trace argument will not work if one assaults the normality of BA directly, instead of proving that B commutes with A^*A.

In order not to tie ourselves down to any particular notion of trace, we formulate the theorem in terms of commutators (a commutator is an expression of the form $PQ - QP$).

Theorem 2. *Suppose A, B, and AB are normal operators on Hilbert space,*

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