DISCRETE POTENTIAL THEORY

BY R. J. DUFFIN

1. **Introduction.** The aim of this paper is to determine in what sense certain well-known properties of the Laplacian differential operator Δ in three dimensions extend to the corresponding difference operator, which may be denoted by D. D operates on functions u defined at points whose rectangular coordinates are integers. The function u may be termed "discrete harmonic" at one of these lattice points if its value there is the mean of its values at the six neighboring points. This is written $Du = 0$. The theory of D may be termed "discrete potential theory."

In physical problems the operator D is often used as an approximation to Δ. There are, however, some problems in which D appears directly, such as random walk and diffusion in a lattice. This concerns the motion of a particle which at each lattice point has an equal probability of jumping to a neighboring lattice point. Another direct application arises if the lattice lines are regarded as metallic wires. This gives an infinite electrical network. If the electric potential of the lattice points is denoted by u, then at every insulated lattice point, $Du = 0$. At a source point, $Du = -w$ where w is the current entering this lattice point. These physical models are of suggestive value for the analysis of D.

The work of Phillips and Wiener [6], Courant, Friedrichs, and Lewy [3], and Petrowsky [5] shows that properties of Δ may be derived from properties of D. It is not clear whether conversely, the properties of D may be derived in any direct fashion from those of Δ. At any rate, in this paper properties of Δ are used merely to suggest corresponding properties of D.

The operator Δ may be treated by the operational calculus based on the Fourier transform. In §2 it is shown that there is a corresponding operational calculus for D based on Fourier series. In particular this calculus shows that there is a function g which satisfies $Dg = 0$ everywhere except at one point where it has a unit source, $Dg = -1$. This function vanishes at infinity and is unique; it is termed the Green's function. The electrical resistance R between two lattice points p and q is related by the formula $R = 2g(p) - 2g(q)$ where p is regarded as the source point of g. Theorem 1 shows that the Green's function decreases steadily in moving away from the source point. Thus the electrical resistance increases steadily to the limiting value $2g(p)$.

An interesting question concerning the Green's function is its asymptotic behavior at large distances from the source point. In §4 it is shown that the

Received July 31, 1952; in revised form, January 15, 1953. Presented to the American Mathematical Society, September 3, 1952. The work on this paper was carried out under Contract No. DA-36-061-ORD-113 with the Office of Ordnance Research, Durham, North Carolina.