DIOPHANTINE EQUATIONS SEPARABLE IN CYCLOTOMIC FIELDS

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1. **Introduction.** This paper generalizes results of two earlier papers [7,8] in which the complete solution in rational integers of certain cubic diophantine equations was obtained. The solutions were represented by integral formulas, *i.e.* formulas involving only rational integers or the indeterminates x_i were given in the form $x_i = F_i/s$ where the F_i are integral formulas and it suffices to restrict s to be the g.c.d. of the F_i. The method for the resolution of these cubic equations hinged on the techniques of separable diophantine equations [2] applied in the algebraic number field $R(e^{2\pi i/3})$, and the solvability of the auxiliary multiplicative equations appearing in the course of the investigation were in the main dependent on the equation $xX\bar{X} = yY\bar{Y}$ where x, y are rational integers and the integers $\bar{X}$, $\bar{Y}$ are the conjugates of X, Y respectively in $R(e^{2\pi i/3})$.

In attempting to generalize these considerations to higher degree cyclotomic fields, and consequently to solve in rational integers the corresponding equations of higher degree, the auxiliary multiplicative equations become complicated and the method of proof used for the multiplicative equations in the cubic case are no longer serviceable. It therefore becomes necessary to develop a straightforward, uniform method to handle with facility the more complicated multiplicative equations. This is provided in §4 and §5.

In these sections we develop an algorithm for the resolution of the multiplicative equation (5.1), which is of a very general type and includes the auxiliary equations appearing in this paper. By means of this algorithm the complete solution of (5.1) can be written down from a knowledge of the solution in multiplicative form of an associated multiplicative system of independent equations in the rational domain. The resolution of this last type of equation is straight forward routine by the method of E. T. Bell [1] although the rapidity with which the number of parameters increases with the degree and number of equal products in a system is disconcerting but inevitable [1; 53].

In §§6, 7, 8 the algorithm is applied to obtain the complete solution of certain diophantine equations of higher degree in the rational domain, for which only partial solutions in some cases have been given hitherto. The equation $x_1^l + x_2^l + \cdots, + x_m^l = 0$ where l is an odd prime is considered and a method is developed which will yield all the integral solutions of this equation when $m \geq 4(l - 2)$.

2. **Notations and properties of the cyclotomic number field.** We introduce the following notations and definitions and also state some of the well-known

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