THE EVALUATION OF RAMANUJAN'S SUM AND GENERALIZATIONS

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1. **Introduction.** A large number of important arithmetical functions $F(k)$ are expressible in the form

$$F(k) = \sum_{d \mid k} f(d)h(k/d), \tag{1}$$

where $f(n)$ and $h(n)$ are arithmetical functions and the sum extends over the divisors of k. It is well-known that the function $F(k)$ will be multiplicative, that is to say $F(1) = 1$ and $F(mn) = F(m)F(n)$ for $(m,n) = 1$, whenever both $f(n)$ and $h(n)$ are multiplicative. In this paper we will discuss more general functions $S(n;k)$ of two integral variables defined by sums of the form

$$S(n;k) = \sum_{d \mid (n,k)} f(d)h(k/d), \tag{2}$$

where now the summation extends over the divisors of the greatest common divisor (n, k) of n and k. When k divides n, the sums (2) specialize to those of the form (1). A famous example of a sum of type (2) is Ramanujan's sum $c_k(n)$ (see [3; 237]) which 'evaluates' the sum of the n-th powers of the primitive k-th roots of unity,

$$c_k(n) = \sum_{\substack{m \bmod k \\ (m,k)=1}} \exp(2\pi i n m/k) = \sum_{d \mid (n,k)} d\mu(k/d), \tag{3}$$

where $\mu(n)$ is the Möbius function and the sum over m is taken over any reduced residue system modulo k.

It is well-known that the sum in (3) is multiplicative in the variable k, $c_k(n)$ $c_{k'}(n) = c_{kk'}(n)$ if $(k,k') = 1$. We will prove that the more general sums (2) are multiplicative in *both* variables, in the sense described in Theorem 1, provided that both $f(n)$ and $h(n)$ are multiplicative. From this it follows that such sums $S(n;k)$ are determined completely in terms of the values obtained when n and k are powers of the same prime. In Theorem 2, suitable specialization of $f(n)$ and $h(n)$ leads to an evaluation of the sums $S(n;k)$ in terms of more familiar functions of the form (1). In particular, this leads to an evaluation of Ramanujan's sum in terms of the Möbius function and Euler's ϕ-function. We then establish a connection between the sums $S(n;k)$ and exponential sums, thus generalizing (3). Finally we discuss certain Dirichlet series whose coefficients involve the numbers $S(n;k)$.

2. **Multiplicative properties.** The sums $S(n;k)$ defined by (2) satisfy the following multiplicative property.

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