BOUNDS FOR DETERMINANTS WITH DOMINANT MAIN DIAGONAL

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1. **Introduction.** Let A be a real matrix with positive main diagonal elements,

$$a_{ii} \geq \sum_{j \neq i} | a_{ij} | \qquad (i = 1, 2, \cdots, n) \tag{1}$$

then the determinant, $[A]$, of A is nonnegative. This is P. Furtwängler's [4] generalization of a theorem which is often called the "Theorem of H. Minkowski, [6], on positive determinants", though it had been proved earlier by L. Lévy [5].

Later papers generalized these results to include determinants with complex elements. If A has complex elements such that

$$| a_{ii} | \geq \sum_{j \neq i} | a_{ij} | \qquad (i = 1, 2, \cdots, n), \tag{2}$$

then $[A]$ is said to be a determinant with dominant main diagonal.

There exist a number of results which give positive bounds for the absolute value of determinants with dominant main diagonals. Recently, R. Oeder [7] proposed a problem which gave a new lower bound for real determinants with positive diagonal elements. His result is a special case of a more general theorem proved a little later by G. B. Price [10].

A. Ostrowski ([8], [9]) has also written several papers on this subject. In his most recent note he has improved and generalized the results of Price.

The bounds given in this paper are different from those mentioned above and are, in many cases, more exact, especially for certain determinants with positive diagonal elements. Many papers have also been written weakening the hypotheses for the nonvanishing of determinants. Among these, some of the sharpest results have been given by Ostrowski, and independently by A. Brauer [1], as a consequence of his theorems on the characteristic roots of a matrix. These results show that, for a matrix with real elements, and positive main diagonal elements, if

$$a_{ii} \cdot a_{kk} \geq \sum_{j \neq i} | a_{ij} | \cdot \sum_{k \neq i} | a_{ik} | \quad (i, k = 1, 2, \cdots, n), \tag{3}$$

then

$$[A] \geq 0, \tag{4}$$

where equality holds in (4) only if it holds for all equations (3). His later papers ([2], [3]) have sharpened these results still further.

In this paper several theorems are given on determinants with only relatively large main diagonals, which do not satisfy (2), and it is shown that these determinants also do not vanish. As a consequence of these theorems, new results are given on bounds for the characteristic roots of a matrix.

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