A CLASS OF MEROMORPHIC FUNCTIONS HAVING THE UNIT CIRCLE AS A NATURAL BOUNDARY

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The continued fractions

$$1 + \overset{\infty}{\underset{n=1}{K}} \left(\frac{d_n z^{\alpha_n}}{1} \right), \tag{1}$$

where α_n is a positive integer and $d_n \neq 0$ for all $n \geq 1$ were first investigated by Leighton and Scott [2]. If the sequences $\{d_n\}$ and $\{\alpha_n\}$ of the continued fraction (1) satisfy the conditions

$$\lim_{n \to \infty} | d_n |^{1/\alpha_n} = 1 \qquad \text{and} \qquad \lim_{n \to \infty} \alpha_n = \infty, \tag{2}$$

then (see [1]) the continued fraction converges to a meromorphic function $f(z)$ for all $| z | < 1$. Scott and Wall [4] were able to show that the function $f(z)$ defined by a continued fraction (1) has the circle $| z | = 1$ as a natural boundary under rather restrictive conditions. Their results are special cases of the following more general theorem recently proved by the author [5]:

The function $f(z)$ represented by the continued fraction (1) *has the circle $| z | = 1$ as a natural bundary provided* (1) *satisfies in addition to* (2) *the conditions: $\alpha_n \geq \alpha_{n-1}$ for all $n \geq 1$ and the sequence $\{\alpha_n\}$ is such that there exists a sequence of positive integers $\{\mu_k\}$ with $\lim \mu_k = \infty$, with the property that for every k there exists an $n(k)$ such that μ_k divides α_n for all $n > n(k)$.*

Using in part methods employed in [5], we establish here the following result.

Theorem. *If the continued fraction* (1) *satisfies condition* (2) *and if in addition*

$$\overline{\lim_{n \to \infty}} \frac{h_n}{h_{n-1}} = \infty,$$

where

$$h_n = \sum_{k=1}^{n+1} \alpha_k,$$

then the function $f(z)$ to which (1) *converges for $| z | < 1$ has the circle $| z | = 1$ as a natural boundary.*

It is clear that this result neither contains nor is contained in the previous one. Both have their interesting aspects. The earlier theorem shows that the sequence $\{\alpha_n\}$ need not increase at a rapid rate, for example $\alpha_n = 2^{[\log n]}$ clearly

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