UNIFORM COMPLETENESS OF SETS OF RECIPROCALS OF LINEAR FUNCTIONS

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1. **Introduction.** In §2 of this paper we give three conditions each of which is necessary and sufficient in order that the set K: $\{(1 + k_p x)^{-1}\}_{p=0}^{\infty}$ ($k_p \neq 0$, $k_p \neq k_q$ if $p \neq q$, and $k_p \not\in [-\infty, -1]$) should be uniformly complete in $F[0, 1]$ (*i.e.*, each continuous function on [0, 1] can be uniformly approximated by linear combinations with numerical coefficients of terms of K). They are (i) the divergence of $\sum_{p=0}^{\infty} (1 - |x_p|)$, where $x_p = [(1 + k_p)^{1/2} - 1]/[(1 + k_p)^{1/2} + 1]$, (ii) if $\{a_p\}_{p=0}^{\infty}$ is a sequence of numbers, then the system of equations $a_p = \int_0^1 (1 + k_p x)^{-1} d\phi(x)$, $p = 0, 1, 2, \cdots$, has at most one solution ϕ in $BV[0, 1]$ ("moment problem"), and (iii) the closed linear manifold generated by K in $F[0, 1]$ should contain the function 1.

In §3 we add the condition that $|\arg (1 + k_p)| \leq \theta < \pi$, $|1 + k_p| \geq \delta > 0$, $p = 0, 1, 2, \cdots$, and prove that the divergence of the series $\sum_{p=0}^{\infty} |k_p|^{-1/2}$ is necessary and sufficient in order that K should be uniformly complete in $F[0, 1]$. Also, we show that if $0 < a < b$ and $K \subset F[0, b]$ then K is uniformly complete in $F[a, b]$.

In the last section we show that in order for K to be uniformly complete in $F[0, 1]$ it is necessary and sufficient that there should exist a function f in $F[0, 1]$ such that the closed linear manifold generated by K in $F[0, 1]$ contain some neighborhood of f. Also, we show that if K is not uniformly complete in $F[0, 1]$, and if we enlarge K by the addition of a finite collection of elements of $F[0, 1]$, then the resulting set is not uniformly complete in $F[0, 1]$.

Szegö [5] proved that if $k_p \to 0$ as $p \to \infty$, then K is uniformly complete in $F[0, 1]$, and Szasz [4] proved that if $k_p \to 0$ as $p \to \infty$, then the set $\{(1 + k_p x)^m\}_{p=0}^{\infty}$, where m is a number not a positive integer or 0, is uniformly complete in $F[0, 1]$. Recently van Herk [6] showed that if k_p is real and positive, $k_p < k_{p+1}$ for $p = 0, 1, 2, \cdots$, and $k_p \to \infty$ as $p \to \infty$, then the divergence of $\sum_{p=0}^{\infty} k_p^{-1/2}$ implies that the moment problem mentioned above (compare (ii)) has at most one solution ϕ in $ND[0, 1]$.

2. **Uniform completeness of K in $F[0, 1]$.** Throughout this section, $\{k_p\}_{p=0}^{\infty}$ denotes a sequence of numbers, distinct from one another and from 0, none of which is a real number less than or equal to -1, and K denotes the sequence $\{(1 + k_p x)^{-1}\}_{p=0}^{\infty}$. If $[a, b]$ is an interval, $F[a, b]$ denotes the collection of all complex-valued continuous functions on $[a, b]$, $BV[a, b]$ the collection of all real-valued functions f of bounded variation on $[a, b]$, such that $f(a) = 0$ and

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