ORDERED ALGEBRAS WHICH CONTAIN DIVISORS OF ZERO

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1. **Introduction.** The following definition of an ordered ring is equivalent to that given by G. Birkhoff [2].

Definition. A ring R is said to be ordered if R contains a subset $R(+)$, called the positive elements of R, satisfying:

(i) for each $x \in R$ exactly one of the following holds:

$$x \in R(+), \qquad x = 0, \qquad \text{or} \qquad -x \in R(+);$$

(ii) $x, y \in R(+)$ implies $x + y \in R(+)$ and $-(xy) \notin R(+)$.

Note that this definition does not impose on an ordered ring the restriction that it contain no proper divisors of zero, a restriction which seems rather severe from an algebraic viewpoint. It is of some interest then to ask whether or not there exist rings ordered in the sense of Birkhoff which contain proper divisors of zero, and if so which rings have these properties. An answer to the first question is easily obtained. Denote by A the nilpotent algebra, over an ordered field F, spanned by e^i, $i = 1, \cdots, n$, with multiplication defined by $e^{n+1} = 0$. If $A(+)$ is the set of all $x = \sum_{i=1}^{n} \xi_i e^i \in A$ for which the first nonzero element in the sequence $\{\xi_i\}$ is a positive element of F, it is easily verified that A is ordered. The second question, as the title of the paper indicates, has been considered only in case the ring is an associative linear algebra of finite dimension over a field.

It should be mentioned at this point that A. A. Albert [1] has shown that an ordered algebra containing no proper divisors of zero is a field. This result is used in proving one of the lemmas for Theorem 2.1, which states in part that an ordered algebra is completely primary provided it is not nilpotent. Theorem 2.1 and its corollaries together with Theorem 3.1 constitute the main results of the present paper. A few comments on ordered nilpotent algebras are contained in the last section.

It may be of interest to mention that for both an ordered nilpotent algebra and an algebra satisfying the hypotheses of Theorem 2.1 the multiplication is continuous in the interval topology. Birkhoff [2] calls this the intrinsic topology. That these algebras are topological rings then follows from the fact that a simply ordered group is a topological group with respect to the interval topology. For a proof of this see Iseki [3].

2. **Ordered algebras which are neither nilpotent nor semi-simple.** The following observation will be used several times and is stated as

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