SUB-BIHARMONIC FUNCTIONS

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1. **Introduction.** The properties of a class of functions called subharmonic are well-known as shown in the work of Riesz [2], Radó [1], and others. A subharmonic function $v(x,y)$, continuous with its partial derivatives of the second order in a domain D, is known to satisfy $\nabla^2 v \geq 0$ at all points of D, where $\nabla^2 v = v_{xx} + v_{yy}$, and any function which satisfies this inequality is subharmonic.

Similarly we may define those functions $u(x,y)$ as sub-biharmonic if

$$\nabla^4 u = \nabla^2(\nabla^2 u) \leq 0 \tag{1}$$

at all points of D.

Riesz has shown that (1) is a necessary and sufficient condition that $\nabla^2 u(x,y)$ be superharmonic. It follows for $\nabla^2 u(x,y) = -g(x,y)$, that $g(x,y)$ is subharmonic. Hence the function $u(x,y)$ which satisfies (1) is termed sub-biharmonic.

The object of this paper is to characterize these functions by certain integral means analogous to the integral means of subharmonic functions. We shall use the following notation:

$$L_\rho(f;x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho,\theta)\, d\theta \tag{2}$$

$$A_\rho(f;x_0,y_0) = \frac{1}{\pi\rho^2}\int_0^{\rho}\int_0^{2\pi} f(\rho,\theta) r\, dr\, d\theta \tag{3}$$

$$\mathfrak{L}_\rho(f;x_0,y_0) = \frac{1}{1-\mu^2}\,[L_{\mu\rho}(f;x_0,y_0) - \mu^2 L_\rho(f;x_0,y_0)] \tag{4}$$

$$\mathfrak{A}_\rho(f;x_0,y_0) = \frac{1}{1-\mu^2}\,[A_{\mu\rho}(f;x_0,y_0) - \mu^2 A_\rho(f;x_0,y_0)], \tag{5}$$

where $0 < \mu < 1$.

2. **Properties of smooth sub-biharmonic functions.** Here we shall consider some integral-mean characterizations of smooth sub-biharmonic functions.

Theorem 1. *A necessary and sufficient condition that $u(x,y)$ be sub-biharmonic at all points of D is that*

$$L_\rho(\nabla^2 u;x_0,y_0) \leq A_\rho(\nabla^2 u;x_0,y_0)$$

for all circles in D.

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