DIFFERENCE EQUATIONS AND J-MATRICES

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1. **Introduction.** Let H denote the (complex) Hilbert space, *i.e.*, the set of all sequences x of (complex) numbers $x_0, x_1, x_2, \cdots$ for which $\| x \|^2 = \sum_0^\infty | x_n |^2 < \infty$, where $\| x \|$ is the customary norm. Corresponding to a complex number λ, a sequence of real numbers $q_0, q_1, q_2, \cdots$, and a sequence of positive numbers $p_0, p_1, p_2, \cdots$, let L_λ denote the linear operation on H defined by $L_\lambda(x_n) = \Delta(p_n \Delta x_n) + (q_n + \lambda)x_{n+1}$, $(n = 0, 1, 2, \cdots)$, where Δ denotes the differencing operation, $\Delta x_n = x_{n+1} - x_n$. The system of equations $L_\lambda(x_n) = 0$ can be considered from two points of view. On the one hand, it is the discrete analogue of the self-adjoint second order differential equation and is amenable to the methods developed by Weyl [19] to obtain the solution theory of that equation. On the other hand, it has the form of a three-termed recursion formula which, together with a real initial condition $ax_0 + bx_1 = 0$, determines a real symmetric infinite matrix of the Jacobi type (*i.e.*, $J = (j_{ik})$, where $j_{ik} = 0$ if $| i - k | > 1$ while $j_{i,i+1} > 0$). A connection between these two viewpoints is implicit in a paper of Hellinger [12].

It is known that the dimension of the closed linear manifold in H which is spanned by the solutions of $L_\lambda(x_n) = 0$ is either 2 for all nonreal λ, or 1 for all nonreal λ. In the first case, it is also 2 for all real λ, and in the second case it is not greater than 1 for any real λ. In the first part of this paper conditions sufficient that the second case hold are obtained in terms of the coefficient sequences $p_0, p_1, p_2, \cdots$, and $q_0, q_1, q_2, \cdots$. These conditions also guarantee that certain associated J-matrices are self-adjoint.

The second part of the paper deals with a sheaf of J-matrices $J(\varphi)$ associated with L_λ by the initial conditions $x_0 \sin \varphi + x_1 \cos \varphi = 0$, where $0 \leq \varphi < \pi$. A theorem of Hartman and Wintner [10], dealing with a more general sheaf of matrices, implies the following results: If the sheaf of matrices $J(\varphi)$ consists of self-adjoint matrices, then every real λ is in the spectrum of at least one of the matrices. The motion of the isolated points of the spectrum of $J(\varphi)$ is monotone and analytic as a function of φ. These results were obtained independently by the author. A proof is given which is simpler and more elementary than that given by Hartman and Wintner for their more general case. The results are extended to cover initial conditions of the form $g(\lambda; t)x_0 + h(\lambda; t)x_1 = 0$, where t is a parameter.

It is known that the section cluster spectrum of a J-matrix (see §7) contains the spectrum of the matrix. Whether or not the two are identical is an unsolved problem. The limit process involved in the passage from the spectrum of the n-th section of a J-matrix to the spectrum of the infinite matrix is investigated.

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