FACTORIZATION OF CERTAIN POLYNOMIALS OVER A COMMUTATIVE RING

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1. **Introduction.** We shall use the word ring to mean commutative ring with at least two elements, and unless otherwise stated every ring considered is assumed to have a unit element 1. If R is a ring, the polynomial ring $R[x_1, x_2, \cdots, x_n]$ in a finite number n of indeterminates will be denoted by $R[x]$. The element f of $R[x]$ is *irreducible over* R if f is not itself a unit of $R[x]$ and any factorization $f = gh$ in $R[x]$ implies that g or h is a unit of $R[x]$. The element f of $R[x]$ may be said to be *primitive* (over R) if the coefficients of terms of f of positive degree generate the unit ideal in R.

Now for certain rings R no primitive polynomial can be irreducible over R. For example, assume that R has nonzero idempotents e_1, e_2 such that $e_1 + e_2 = 1$, and therefore $e_1e_2 = 0$. If c_i, d_i are elements of the ring $e_iR[x]$ such that $c_id_i = e_i (i = 1, 2)$, then clearly $f = (c_1 + c_2f)(d_1f + d_2)$. Furthermore, neither of these factors can be a unit. For if, for example, $c_1 + c_2f$ has an inverse k in $R[x]$, then $c_2fk = e_2$, and c_2f is a unit of the ring $e_2R[x]$. This implies ([2; 5] or [4; 683]) that every coefficient of a term of positive degree in c_2f is nilpotent. However, this is impossible since f is a primitive polynomial and c_2 is not a divisor of zero in $e_2R[x]$.

We shall see that for certain polynomials, the *only* factorizations into nonunit factors are of the kind just described. For convenience, we now make the following definition.

The element f of $R[x]$ is *weakly irreducible over* R if to each factorization $f = gh$, where g and h are nonunits of $R[x]$, there exist nonzero idempotents e_1, e_2 of R with $e_1 + e_2 = 1$, and elements c_i, d_i of $e_iR[x]$ with $c_id_i = e_i (i = 1, 2)$, such that $g = c_1 + c_2f$ and $h = d_1f + d_2$.

Clearly, a polynomial which is irreducible over R is weakly irreducible over R. Also, if R is a ring which has no idempotents except 0 and 1, a polynomial which is weakly irreducible over R is necessarily irreducible over R.

The main result of this note is Theorem 1 which gives conditions under which a primitive polynomial is weakly irreducible.

An interesting special case arises if f is a polynomial with *integral* coefficients, in which case f may be considered as an element of $R[x]$, where R is *any* ring. Theorem 2 states that if f is such a polynomial which is irreducible over every *field*, then f is weakly irreducible over every ring. Classical examples of polynomials irreducible over every field are the determinants of any order with indeterminate elements, and the resultant of two polynomials in one indeterminate with indeterminate coefficients.

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