ENTIRE FUNCTIONS AND TRIGONOMETRIC POLYNOMIALS

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1. **Introduction.** An entire function $f(z)$ is said to be of exponential type if there is a positive number γ such that

$$(1) \qquad f(z) = O(e^{\gamma |z|})$$

uniformly in all directions as $|z|$ becomes large. It was shown by S. Bernstein [2] in 1923 that if an entire function of exponential type γ is bounded by 1 on the real axis then its first order derivative is bounded by γ on the real axis; and this bound is clearly precise, as shown by the example $f(z) = \sin \gamma z$. The class of entire functions to be considered in the present note are those which are bounded in a given closed set E of the real axis, and it will be supposed that for some positive numbers A, α each interval of the real axis of length A contains a subset of E of measure greater than α. It is to be shown that at almost all points x of E there is a number $c = c(x)$ such if $f(z)$ is an entire function of exponential type γ which is bounded by 1 in E then

$$(2) \qquad |f'(x)| \leq \gamma c.$$

The interest of this result is that with a given set E, inequality (2) can be inferred for entire functions of every exponential type $\gamma > 0$. There are much milder conditions on the set E which will imply that inequality (2) is satisfied whenever γ is less than some fixed number γ_0, but fail completely for $\gamma > \gamma_0$.

A subclass of the entire functions of exponential type which is of special interest is the class of trigonometric polynomials

$$(3) \qquad T_n(z) = \sum_{\nu=0}^{n} (a_\nu \cos \nu z + b_\nu \sin \nu z).$$

Since $T_n(z)$ is an entire function of exponential type n it follows from a theorem previously mentioned that if a trigonometric polynomial is bounded by 1 for real z then its first order derivative is bounded by n for real z, a result proved by S. Bernstein [1] in 1912. As a possible extension of Bernstein's theorem on trigonometric polynomials, Privaloff [7] stated in 1916 that if H is a closed set of positive measure on the interval $(-\pi, \pi)$ then for each $\epsilon > 0$ there is a subset H^Δ of H having measure greater than $|H| - \epsilon$ and a number $B(\epsilon)$ such that if $T_n(x)$ is a trigonometric polynomial of degree n which is bounded by 1 in H then

$$(4) \qquad |T_n'(x)| \leq nB(\epsilon)$$

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