THE CRITICAL POINTS OF CERTAIN POLYNOMIALS

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1. **Introduction.** In this note we study the location of the critical points of polynomials of the form $p(x) = (x^2 + 1)^k (x - \alpha_1) \cdots (x - \alpha_l)$ where the α_i are real. A root of the derivative $p'(x)$ will be called a trivial critical point if it arises from a multiple root of $p(x)$; the other roots of $p'(x)$ will be called nontrivial critical points. All but two of the roots of $p'(x)$ can be accounted for by the multiplicities of roots of $p(x)$ and by Rolle's theorem. For $k \leq 4$, the theorem on page 32 of [1] shows that if the roots α_i lie in the closed interval $[0,(n(n - 2k))^{\frac{1}{2}}/k]$ and are not all concentrated at $(n(n - 2k))^{\frac{1}{2}}/k$ then the two critical points not accounted for are nonreal (here $n = 2k + l$, the degree of $p(x)$). Our theorem (§4 below) gives the general situation in which there are nontrivial nonreal critical points.

In §§2 and 3 we prove some preliminary results which are used in the proof of the main theorem. For notations and concepts not explicitly defined we refer the reader to [1].

2. **Preliminary lemmas.** Let $\mathfrak{F}(\delta,\gamma)$ be the class of polynomials with roots of multiplicity k at i and $-i$ and with l real roots α_i such that $\delta \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_l \leq \gamma$.

Lemma 1. *Suppose $f_1 \in \mathfrak{F}(\delta, \gamma)$ is such that between two distinct successive roots α_m, α_{m+1} of f_1 there are three real critical points r_1, r_2, and r_3. Then there exists an $f_2 \in \mathfrak{F}(\delta, \gamma)$ of the form $f_2 = (x^2 + 1)^k (x - \alpha)^m (x - \beta)^{l-m}$ with three real critical points $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$ and $\delta \leq \alpha < \bar{r}_1 \leq \bar{r}_2 \leq \bar{r}_3 < \beta \leq \gamma$.*

The nontrivial critical points of $f_1(x)$ are given by the roots of the logarithmic derivative $F_1(x)$ of $f_1(x)$. We write $F_1(x)$ in the form

$$(1)\quad F_1(x) = \frac{2kx}{1 + x^2} + \frac{1}{x - \alpha_1} + \cdots + \frac{1}{x - \alpha_m} + \frac{1}{x - \alpha_{m+1}} + \cdots + \frac{1}{x - \alpha_l}.$$

By assumption, $F_1(r_1) = F_1(r_2) = F_1(r_3) = 0$ where $\alpha_m < r_1 \leq r_2 \leq r_3 < \alpha_{m+1}$. Since F_1 is positive immediately to the right of α_m and negative immediately to the left of α_{m+1} and r_1, r_2, and r_3 are the only roots of F_1 in $[\alpha_m, \alpha_{m+1}]$, it is clear that $dF_1(r_1)/dx \leq 0$, $dF_1(r_2)/dx \geq 0$, $dF_1(r_3)/dx \leq 0$.

Let α and β be the unique roots of the equations

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