THE VARIATION OF THE SPECTRUM OF A NORMAL MATRIX

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If A and B are two normal matrices, what can be said about the "distance" between their respective eigenvalues if the "distance" between the matrices is known? An answer is given in the following theorem (in what follows, all matrices considered are $n \times n$; the Frobenius norm $\| K \|$ of a matrix K is $(\sum_{ij} | k_{ij} |^2)^{1/2}$).

Theorem 1. *If A and B are normal matrices with eigenvalues $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$ respectively, then there exists a suitable numbering of the eigenvalues such that $\sum_i | \alpha_i - \beta_i |^2 \leq \| A - B \|^2$.*

Proof. Let A_0 and B_0 denote the diagonal matrices with diagonal elements $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$ in arbitrarily fixed order. Since A and B are normal, there are unitary matrices U and V such that $A = UA_0U^*$ and $B = UVB_0V^*U^*$. Then we have $\| A - B \| = \| A_0 - VB_0V^* \|$; hence, Theorem 1 is equivalent to

(1) *The minimum of $\| A_0 - VB_0V^* \|^2$, where V ranges over the set of all unitary matrices, is attained for $V = P$, where P is an appropriate permutation matrix.*

To prove (1), observe that

$$\begin{aligned} \| A_0 - VB_0V^* \|^2 &= \text{Trace}\,(A_0 - VB_0V^*)(A_0^* - VB_0V^*) \\ &= \text{Trace}\,(A_0A_0^* + B_0B_0^*) + r(V), \end{aligned}$$

where $r(V) = \sum_{ij} d_{ij}w_{ij}$; $d_{ij} = -(\alpha_i\bar{\beta}_j + \bar{\alpha}_i\beta_j)$, $w_{ij} = v_{ij}\bar{v}_{ij}$, $V = (v_{ij})$. Hence, min $\| A_0 - VB_0V^* \|^2$ is attained at a V for which $r(V)$ is a minimum.

Let $\mathfrak{X}_n$ be the set of all matrices $X = (x_{ij})$ such that

$$(2) \qquad \sum_i x_{ij} = 1, \qquad \sum_j x_{ij} = 1, \qquad x_{ij} \geq 0 \qquad (i, j = 1, \cdots, n).$$

Let $\mathfrak{W}_n$ be the set of all matrices $W = (w_{ij}) = (v_{ij}\bar{v}_{ij})$, with $V = (v_{ij})$ a unitary matrix. Then $\mathfrak{W}_n$ is a subset of $\mathfrak{X}_n$ (indeed, $\mathfrak{W}_n$ is a proper subset, if $n \geq 3$, in view of

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