THE PRINCIPLE OF CONDENSATION OF SINGULARITIES

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1. Introduction and results. 1.1. Two general theorems have been proved in a joint paper of S. Banach and H. Steinhaus [1] which are known as *principle of uniform boundedness* and *principle of condensation of singularities.* The common proofs of these principles are based upon a category argument, originally due to S. Saks. In a recent paper [2] we gave a new proof of the first principle using a method of H. Lebesgue [4] instead of the categories. As we pointed out there, the advantage of this proof is that it holds also in a general case of operations, where the category argument fails. It is the purpose of the present paper to show that using similar methods as in [2] the condensation-principle can also be extended to the same generality.

Let us consider an operation $u(x)$ defined over a complete vector space E into a normed vector space E'. We say that $u(x)$ is *bounded and homogeneous* if there exists a positive $M > 0$ such that $|| u(x) || \leq M || x ||$ and if

$$|| u(\lambda x) || = | \lambda | \cdot || u(x) || \tag{1}$$

for every real λ. Then it is clear that the *norm* of the operation $u(x)$ can be defined in the same way as in the case of linear operations, *i.e.*,

$$| u | = \operatorname*{supr}_{||x||=1} || u(x) || = \operatorname*{supr}_{x \neq \theta} || u(x) ||/|| x ||. \tag{2}$$

Obviously we have $|| u(x) || \leq | u | \cdot || x ||$ for every $x \in E$ and this inequality is the best possible. In our preceding paper we have introduced the concept of *asymptotically subadditive* sequences of bounded and homogeneous operations as follows:

Definition. *We say that the sequence* $\{u_n(x)\}$; $n = 1, 2, \cdots$ *of the bounded and homogeneous operations* $u_n(x)$; $x \in E$ *is asymptotically subadditive if*

$$|| u_n(x + y) || \leq || u_n(x) || + O(| u_n | \cdot || y ||) \tag{3}$$

uniformly in $x, y \in E$; $|| x ||, || y || \leq 1$ as $n \to \infty$, *furthermore if*

$$\inf_{||y|| \leq 1} [|| u_n(x + y) || + || u_n(x) || - || u_n(y) ||] \geq o(| u_n |) \tag{4}$$

for every $x \in E$; $|| x || \leq 1$ *but not necessarily uniformly in* x. (*In other words, the left hand side of* (4) *can be negative, but it is greater than* $-c(n; x) \cdot | u_n |$ *where* $c(n; x) \to 0$ *as* $n \to \infty$ *for every* $x \in E$).

1.2. Now we can state the generalized principle of condensation in the following way:

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