FUNCTIONAL INEQUALITIES IN THE ELEMENTARY THEORY OF PRIMES

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1. **Introduction.** In what follows

$$a > 1, \quad \delta > 0, \quad A_1 > 0, \quad A_2 \geq 0, \quad A_3 > 0, \quad \chi(x) > 0;$$

$f(x)$ and $\chi(x)$ are real functions, bounded and integrable in every finite interval $a \leq x \leq X$; and $\chi(x) = o(x)$ as $x \to \infty$. We suppose $f(x)$ to satisfy the three inequalities

$$|f(x_2) - f(x_1)| \leq A_1 |x_2 - x_1| + A_2 x_1^{-\delta}, \tag{1.1}$$

$$\left| \int_{x_1}^{x_2} f(y)\, dy \right| \leq A_3, \tag{1.2}$$

$$x\,|f(x)| \leq \int_a^x |f(y)|\, dy + \chi(x) \tag{1.3}$$

for all $x, x_1, x_2 \geq a$.

The inequality (1.1) ensures that $f(x)$ does not change value too rapidly, but does not prevent discontinuities. By (1.2) either

$$\int^{\infty} |f(y)|\, dy < \infty$$

or the positive and negative values of $f(x)$ roughly offset one another. (1.3), the most interesting inequality of the three, provides that $|f(x)|$, apart from a term which is $o(1)$, is less than the average of $|f(y)|$ for $a \leq y \leq x$. We shall see (Theorem 1 (i) below) that it follows, though not trivially, from (1.1), (1.2) and (1.3) that $f(x) \to 0$ as $x \to \infty$.

My object is to determine, as precisely as possible, the way in which the order of $f(x)$ depends on that of $\chi(x)$. I shall show that, in a certain sense, my results are best possible. I prove

Theorem 1. *Let $f(x)$ satisfy* (1.1), (1.2) *and* (1.3). *As $x \to \infty$,*

(i) $f(x) \to 0$;

(ii) *if* $\chi(x) = O\{x(\log x)^{-\frac{3}{2}}\}$, *then* $f(x) = O\{(\log x)^{-\frac{1}{2}}\}$;

(iii) *if* $\chi(x) = O(x\phi^3)$, *where* $\phi = \phi(x)$ *is positive and decreases steadily* (*in the non-strict sense*) *to* 0 *and* $\phi(x)(\log x)^{\frac{1}{2}}$ *is non-decreasing, then* $f(x) = O(\phi)$.

Clearly (ii) is the special case of (iii) when $\phi(x) = (\log x)^{-\frac{1}{2}}$. Again (i) is a corollary of (iii), though we in fact prove (i) first. The restriction that $\phi(x)$ $(\log x)^{\frac{1}{2}}$ is non-decreasing ensures that $\phi > A(\log x)^{-\frac{1}{2}}$.

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